VARIOUS DOMAIN CONSTANTS RELATED TO THE UNIFORM PERFECTNESS

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Abstract. We gather comparison theorems for various domain contants related to the uniform perfectness, introduced by Pommerenke. Among these theorems, our main result is a comparison of modulus and injectivity radius of a hyperbolic Riemann surface. Its proof depends on a comparison of extremal and hyperbolic lengths, which seems to be interesting in itself. We also discuss the invariance (or, variance) of these domain constants under the conformal, quasiconformal or Möbius maps.

1. Introduction

A simply connected plane domain $D \subset \mathbb{C}$ with the hyperbolic metric $\rho_D = \rho_D(z)|dz|$ of constant curvature -4 has known to have many remarkable geometric or analytic properties. For instance, the hyperbolic density $\rho_D(z)$ is uniformly estimated by $\delta_D(z) = \mathrm{dist}(z, \partial D)$:

$$\frac{1}{4} \leq \rho_D(z)\delta_D(z) \leq 1, \tag{1.1}$$

and the Schwarzian derivative S_f of the Riemann map $f : \mathbb{H} \to D$ is hyperbolically bounded (by a uniform bound):

$$\|S_f\|_{\mathbb{H}} = \sup_{z\in\mathbb{H}} |S_f(z)|(2\mathrm{Im}z)^2 \leq 6. \tag{1.2}$$

In [3], it was first recognized that the similar property as (1.1) is also valid for a wider class of domains, i.e., modulated domains. While the simple connectedness of a plane domain is characterized by the connectedness of its complement, Pommerenke had an insight into the fact that the above class of domains is also characterized by a way of condensation of its boundary points, called the uniform perfectness by him. Pommerenke thoroughy investigated uniformly perfect sets in [29] and [30], and found so many equivalent definitions for uniform perfectness. Moreover he and

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other authors observed that the limit set of any non-elementary finitely generated Kleinian group and the Julia set of any rational map with degree greater than one are uniformly perfect ([30] and [20], [13]). After the works of Pommerenke, many authors made further investigations of uniform perfectness (see also [1], [5], [8], [9], [10], [18], and [19]).

These facts tell the richness of the notion of uniform perfectness (or equivalently, modulatedness). But, in spite of many studies, explicit comparisons have not been completely made for those domain constants relating uniform perfectness (or, modulatedness), at least in the literature with some exceptions ([10], [25], etc.).

In this article, we try to gather explicit estimates for those domain constans, most of which are, or may be known as results in quite different contexts. We also refer to the invariance or quasi-invariance of those constants under Möbius, conformal, or quasiconformal mappings.

The contents of this article consist of the following. Section 2 is devoted to give a precise definition for those domain constants related to uniform perfectness, and to present explicit estimates for these constants. Especially, we describe the injectivity radius and the modulus of a given Riemann surface by hyperbolic and extremal lengths, respectively, and from this we deduce our main estimate (Theorem 2.3) as an immediate corollary of the results in sec tions 5 and 6.

We discuss in Section 3 the invariance (or, variance) of those domain constants under some class of functions (Möbius, conformal, or quasiconformal mappings). In Section 4, as a special case, we observe the geometry of round annuli, which is fundamental for estimation and gives examples with exactly computable domain constants.

Recently, Riemann surfaces are studied by using extremal lengths of (homotopy class of) simple closed curves, as well as hyperbolic lengths (see, for instance, [7], [15], [26], [34], and [38]). We establish, in Section 5, a comparison theorem for hyperbolic and extremal lengths, which seems to be useful in itself for the geometry of Riemann surfaces as above. In Section 6, we explain that integrable holomorphic quadratic differentials are also (hyperbolically) bounded for modulated Riemann surfaces, and as a corollary of this observation we obtain quite different estimate (Corollary 6.3) from the one obtained by the collar lemma. The ideas in this section are heavily indebted to Matsuzaki [22]. Proofs of several results in this article are postponed to Section 7, where straightforward but slightly boresome calculations are performed.

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2. Various domain constants related to uniform perfectness

At first, we consider an arbitrary hyperbolic Riemann surface R, that is, there exists a holomorphic universal covering map $p : \mathbb{H} \to R$ of R from the upper half plane $\mathbb{H} = \{z \in \mathbb{C}; \mathrm{Im} z > 0\}$. Let Γ be the covering transformation group of $p : \mathbb{H} \to R$, then Γ is a torsion-free discrete subgroup of $Aut\mathbb{H} = PSL(2, \mathbb{R}) = \mathrm{Möb}(\mathbb{H})$. Since the Poincaré (or, hyperbolic) metric $\rho_{\mathbb{H}}(z)|dz| = \frac{|z|}{2\mathrm{Im} z}$ on $\mathbb{H}$ is Möbius invariant, there exists the unique Riemannian metric $\rho_R = \rho_R(z)|dz|$ such that $\rho_{\mathbb{H}} = p^*\rho_R := \rho_R(p(z))|p'(z)||dz|$. The metric ρ_R is called also the hyperbolic metric of R. Some authors prefer to use $\frac{|dz|}{\mathrm{Im} z}$ as the Poincaré metric instead of $\frac{|dz|}{2\mathrm{Im} z}$, so we should be careful of difference caused by the factor 2 when reading the papers of such authors. We denote by $D_R(q, r)$ the hyperbolic disk $\{x \in R; d_R(x, q) < r\}$ centered at $q \in R$ with radius $r > 0$, where d_R denotes the hyperbolic distance naturally defined by ρ_R. Let $\iota_R(q)$ denote the injectivity radius at $q \in R$, i.e.,

$$\begin{aligned}\iota_R(q) &= \sup\{r > 0; D_R(q, r) \text{ is simply connected }\} \\ &= \sup\{r > 0; p \text{ is injective in } D_{\mathbb{H}}(\tilde{q}, r)\},\end{aligned}$$

where $\tilde{q}$ is a point of $\mathbb{H}$ satisfying $p(\tilde{q}) = q$. Further, we call $I_R := \inf_{q \in R} \iota_R(q)$ the *injectivity radius* of R.

Hyperbolic length. Here we explain that I_R has several geometric meanings. We denote by $\mathcal{C} = \mathcal{C}_R$ the set of all free homotopy classes $[\alpha]$ of nontrivial closed curves α on R, and let $\mathcal{S} = \mathcal{S}_R$ the subset of $\mathcal{C}$ consisting of homotopy classes represented by simple closed curves.

For $[\alpha] \in \mathcal{C}$, we define the hyperbolic length $\ell[\alpha] = \ell_R[\alpha]$ of $[\alpha]$ by

$$\ell[\alpha] = \inf_{\alpha' \simeq \alpha} \int_{\alpha'} \rho_R(z)|dz|. \tag{2.1}$$

Then, as is easily seen (cf. [18] Theorem 3), it holds that

$$I_R = \inf_{[\alpha] \in \mathcal{C}} \frac{\ell[\alpha]}{2} = \inf_{[\alpha] \in \mathcal{S}} \frac{\ell[\alpha]}{2}. \tag{2.2}$$

On the other hand, when $\gamma \in \Gamma$ covers a closed curve α on R, we directly see that $\ell[\alpha]$ coincides with the *translation length* l_γ of γ, that is

$$|\mathrm{tr}\gamma| = 2\cosh\frac{\ell[\alpha]}{2},$$

where $|\mathrm{tr}\gamma| = |a + d|$ if γ is represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$. In particular, we have

$$\inf_{\gamma \in \Gamma \setminus \{1\}} |\mathrm{tr}\gamma| = 2\cosh I_R.$$

Modulus. An annulus (= doubly connected planar domain) A contained in R is called *essential* if the natural homomorphism $\iota_* : \pi_1(A, *) \to \pi_1(R, *)$ induced by the inclusion map $\iota : A \hookrightarrow R$ is injective. We denote by $\mathcal{A}_R$ the totality of essential annuli in R, and set

$$M_R := \sup_{A \in \mathcal{A}_R} m(A),$$

where $m(A)$ is the modulus of A which is defined in this article by the number m such that A is conformally equivalent to the round annulus $\{z \in \mathbb{C}; 1 < |z| < e^m\}$. (Conventionally, we define $M_R = 0$ if $\mathcal{A}_R = \emptyset$, i.e., R is simply connected.) We call M_R the *modulus* of R, and we say that R is *modulated* if $M_R < +\infty$ (cf. [6]).

Remark. Here again, we should be cautious about the definition of the moduli (or, modules) of annuli. For instance, in many literatures, the modulus of the annulus $\{z \in \mathbb{C}; 1 < |z| < e^m\}$ is defined by $\frac{m}{2\pi}$ instead of m.

The domain constant M_R also has a significant expression by extremal lengths similar to (2.2). Now we explain this. For $[\alpha] \in \mathcal{C}$, we define the *extremal length* $E[\alpha] = E_R[\alpha]$ of $[\alpha]$ by

$$E[\alpha] = \sup_{\tau} \frac{\left(\inf_{\alpha' \in [\alpha]} \int_{\alpha'} \tau(z)|dz|\right)^2}{\iint_R \tau(z)^2 dxdy}$$

where the supremum is taken over all Borel measurable conformal metrics $\tau = \tau(z)|dz|$ on R.

The next theorem, which claims the existence of an extremal conformal metric τ_0 attaining the above supremum, is fundamental for our method.

Theorem 2.1 (Jenkins and Strebel, cf. [14] **and** [32]**).** *For any $[\alpha] \in \mathcal{S}_R$, there exists an integrable holomorphic quadratic differential φ_0 (called the Jenkins-Strebel differential for $[\alpha]$ with closed horizontal trajectries homotopic to α) and its characteristic ring domain $R_0 \in \mathcal{A}_R$ satisfies the following conditions.*

(1)
$$E[\alpha] = \frac{\left(\inf_{\alpha' \in [\alpha]} \int_{\alpha'} |\varphi_0|^{1/2}|dz|\right)^2}{\iint_R |\varphi_0| dxdy},$$
i.e., $\tau_0 = |\varphi_0|^{1/2}$ is an extremal metric,

(2) $E[\alpha] = \frac{2\pi}{m(R_0)}$,

(3) $m(R_1) \leq m(R_0)$ *for all $R_1 \in \mathcal{A}_R$ such that the core curve is homotopic to α.*

In particular, by observing this theorem, we can readily see the following

Corollary 2.2.

$$\inf_{[\alpha]\in\mathcal{S}_R} E[\alpha] = \frac{2\pi}{M_R}.$$

One of our main results in this article is the next theorem, which connects the quantities I_R and M_R.

Theorem 2.3. *Let R be a hyperbolic Riemann surface with the hyperbolic metric ρ_R of constant curvature -4. Then, we have the following estimate.*

$$2I_R \le \frac{\pi^2}{M_R} \le \min\{2I_R e^{2I_R}, 2{I_R}^2 \coth^2 I_R\}. \tag{2.3}$$

Here, the equality occurs in the left hand side if and only if R is a doubly connected planar Riemann surface or $I_R = 0$.

Remarks. By a numerical calculation, we see that $2I_R e^{2I_R} > 2I_R^2 \coth^2 I_R$ if and only if $I_R > 0.45752\cdots$.

For a triply connected (planar) Riemann surface, we have a better estimate (see Corollary 5.5).

We shall prove this theorem by dividing it into three parts as follows:

$$2I_R \le \frac{\pi^2}{M_R}, \tag{2.4}$$

$$\frac{\pi^2}{M_R} \le 2I_R e^{2I_R}, \tag{2.5}$$

$$\frac{\pi^2}{M_R} \le 2{I_R}^2 \coth^2 I_R. \tag{2.6}$$

The estimates (2.4) and (2.5) are obtained by Corollary 5.3 in Section 5 combined with (2.2) and Corollary 2.2. In fact, by Theorem 5.2 we can show a slightly sharper result than (2.5):

$$\frac{\pi^2}{M_R} \le \frac{\pi I_R}{\arctan\left(\frac{1}{\sinh 2I_R}\right)}.$$

A proof of (2.6) is given in Section 6. By virture of (2.6), we observe that

$$\frac{1}{M_R} = O({I_R}^2) \quad \text{as} \quad I_R \to \infty. \tag{2.7}$$

Thus the estimate (2.6) is much better than (2.5) when I_R is sufficiently large, while (2.5) is very sharp when I_R is sufficiently small. The author does not know whether the exponent 2 in (2.7) is best possible or not.

Hahn metric. Now we introduce the Hahn metric $\hat{\rho}_R$ on R. For a point $q \in R$ and a complex coordinate z of R around q, $\hat{\rho}_R = \hat{\rho}_R(z)|dz|$ is defined by

$$\hat{\rho}_R(z)|dz| = \inf_G \rho_G(z)|dz|,$$

where G ranges over all simply connected domain in R containing q. Then, $\hat{\rho}_R$ is a continuous Riemannian metric on R and called the *Hahn metric* on R (for more informations, see [11], [24] and [10]). By definition, the Hahn metric is conformally invariant, and since $\rho_G \geq \rho_R$ on $G \subset R$ by the Schwarz-Pick lemma, it follows that $\hat{\rho}_R \geq \rho_R$. Noting that $\rho_R/\hat{\rho}_R$ is a (well-defined) continuous function on R and is not greater than 1, we set

$$K_R := \inf_{q \in R} \frac{\rho_R}{\hat{\rho}_R}(q).$$

The function $\rho_R/\hat{\rho}_R$ is known to be comparable to ι_R.

Theorem 2.4 (Gotoh [10]**).**

$$\tanh \iota_R(q) \leq \frac{\rho_R}{\hat{\rho}_R}(q) \leq 4 \tanh \iota_R(q).$$

Corollary 2.5.

$$\tanh I_R \leq K_R \leq 4 \tanh I_R.$$

In the rest of the present section, we shall consider only hyperbolic plane domains D, i.e., subdomains of $\widehat{\mathbb{C}}$ excluding more than two points. Therefore the representation $\rho_D(z)|dz|$ of the hyperbolic metric of D by the natural global coordinate z gives a global function $\rho_D(z)$ on D which is called the *hyperbolic* (or, *Poincaré) density* of D. For $z \in D \setminus \{\infty\}$, we denote by $\delta_D(z)$ the distance from z to the boundary ∂D, that is, $\delta_D(z) = \inf_{w \in \partial D} |z - w|$.

Schwarzian derivative. Let $p : \mathbb{H} \to D$ a holomorphic universal covering, then the Schwarzian derivative of p can be defined to be $S_p = (\frac{p''}{p'})' - \frac{1}{2}\frac{p''}{p'})^2$. Similarly as the case that D is simply connected, we consider the hyperbolic sup norm of the Schwarzian derivative:

$$N_D := \|S_p\|_{\mathbb{H}} = \sup_{z \in \mathbb{H}} |S_p(z)| \rho_{\mathbb{H}}(z)^{-2} = \sup_{z \in \mathbb{H}} |S_p(z)| (2\mathrm{Im} z)^2.$$

This quantity has a close connection with the injectivity radius. In fact, for a non-simply connected domain D, an estimate:

$$2 \coth^2 I_D \leq N_D \leq 6 \coth^2 I_D$$

has been noted essentially by [31], [16], [37], [17] (see also [33]). Later, Minda and Ma improved the estimate as the following.

Theorem 2.6 ([25] **and** [19]). *Let D be a hyperbolic subdomain of $\widehat{\mathbb{C}}$. If D is not simply connected, we have*

$$2+\frac{\pi^2}{2I_D^2}\le N_D\le 6(1-1/M)\coth^2 I_D, \tag{2.8}$$

where $M>1$ is the number satisfying

$$\exp\left(2I_D\sqrt{1+3(1-1/M)\coth^2 I_D}\right)-2\sqrt{(\tanh^2 I_D+3)M-3}-1=0.$$

Remark. As is stated in the introduction, if D is simply connected, a sharp estimate $N_D\le 6$, known as tha Nehari-Kraus theorem, is valid. Moreover, Nehari [27] showed that if $N_D\le 2$ then D must be simply connected.

Round modulus. A subset A of the form $\{z\in\mathbb{C}; r_1<|z-a|<r_2\}\quad(0\le r_1<r_2\le\infty)$ is called a *round annulus* with center a, written by Cent$A$, and of modulus $m(A)=\log r_2/r_1$. Moreover, if $m(A)<\infty$ the curve $\{z;|z-a|=\sqrt{r_1r_2}\}$ is the unique simple closed geodesic of A, called the *core curve* and denoted by Core$A$. For a hyperbolic plane domain $D\subset\widehat{\mathbb{C}}$, we denote by $\mathcal{A}_D^\circ$ the set of all essential round annuli in D, and set

$$M_D^\circ:=\sup_{A\in\mathcal{A}_D^\circ} m(A).$$

We call M_D° the *round modulus* of D. Since $\mathcal{A}_D^\circ\subset\mathcal{A}_D$, we have $M_D^\circ\le M_D$. On the other hand, there exists a domain D such that $M_D>0$ and $M_D^\circ=0$. For example, consider a domain $D=\{x+iy\in\mathbb{C};|x|<2 \text{ and } |y|<\varepsilon\}\setminus[-1,1]$ for sufficiently small $\varepsilon>0$. Nevertheless, some kind of reverse inequality holds.

Theorem 2.7 (cf. McMullen [23]). *If a hyperbolic plane domain D dose not contain ∞, we have*

$$M_D\le M_D^\circ+5\log 2=M_D^\circ+3.4657\cdots.$$

In this theorem, the constant $5\log 2$ seems to be best possible. For the case $\infty\in D$, we also have a similar result, which is obtained by the above theorem combined with Theorem 3.1.

Theorem 2.8. *For any hyperbolic plane domain $D\subset\widehat{\mathbb{C}}$, we have the following estimate.*

$$M_D\le 2M_D^\circ+9\log 2-2\log 3=2M_D^\circ+4.0411\cdots.$$

Furthermore, we can define the round modulus $\mu_D^\circ(z)$ at z for each point z of D by

$$\mu_D^\circ(z)=\sup\{m(A);A\in\mathcal{A}_D^\circ \text{ and } z\in \mathrm{Core}A\},$$

and we set $\mu_D^\circ(z)=0$ if $\{A\in\mathcal{A}_D^\circ; z\in\mathrm{Core}A\}$ is empty. (For this quantity, see [23].) Of course, we have $M_D^\circ=\sup_{z\in D}\mu_D^\circ(z)$.

Capacity density. First, we define an auxiliary constants $E_D, E'_D(\leq 1)$ by

$$E_D = \sup\{E > 0; \{z \in \partial D; Er < |z-a| < r\} \neq \emptyset \text{ for all } a \in \partial D \text{ and } 0 < r \leq \mathrm{diam}\partial D\},$$

$$E'_D = \inf\{e^{-m(A)}; A \in \mathcal{A}^\circ_D \text{ with } \mathrm{Cent}A \in \partial D\}.$$

If $\{A \in \mathcal{A}^\circ_D; \mathrm{Cent}A \in \partial D\}$ is empty, we set $E'_D = 1$. Here, we observe the next fact.

Lemma 2.9. *If $E_D < \frac{1}{2}$ or $\mathrm{diam}\partial D = \infty$, then it is valid that $E_D = E'_D$. Otherwise ($E_D \geq \frac{1}{2}$), we can only say that $E_D \leq E'_D$. Thus, in any case, we have*

$$E_D \leq E'_D \leq 2E_D.$$

Remark. For instance, in the case that $\partial D = [-1, 1]$, we see that $E_D = \frac{1}{2}$ while $E'_D = 1$.

Proof. If $A = \{Er < |z-a| < r\} \in \mathcal{A}^\circ_D$ satisfies that $a \in \partial D$, then $r \leq \mathrm{diam}\partial D$ since A separates ∂D. This means that $E_D \leq E = e^{-m(A)}$, thus we see that $E_D \leq E'_D$. On the other hand, for $a \in \partial D$ and $0 < r < \mathrm{diam}\partial D$, suppose that $A = \{Er < |z-a| < r\}$ does not intersect ∂D. If $A \notin \mathcal{A}^\circ_D$, then ∂D must be contained in $\{z; |z| \leq Er\}$, therefore $\mathrm{diam}\partial D \leq 2Er$, which implies that $\mathrm{diam}\partial D < \infty$ and that $E > \frac{1}{2}$. Now the desired conclusion follows. □

Further we have the following elementary

Lemma 2.10.

$$\log \frac{1}{E'_D} \leq M^\circ_D \leq \log \frac{E'_D + 2}{E'_D} \leq \log \frac{1}{E'_D} + \log 3.$$

Proof. The left hand side inequality is trivial by definition. Let $A = \{Er < |z-b| < r\} \in \mathcal{A}^\circ_D$ (here, b does not necessarily belong to ∂D). Since A separates ∂D, there exists a point a in ∂D such that $|a-b| \leq Er$. Let $A' = \{Er + |a-b| < |z-a| < r - |a-b|\}$ then $A' \subset A$ and A' separates ∂D, too. This implies that

$$E'_D \leq \frac{Er + |a-b|}{r - |a-b|} \leq \frac{2Er}{r - Er} = \frac{2E}{1-E},$$

i.e., $m(A) = \frac{1}{E} \leq \frac{E'_D+2}{E'_D}$. □

Now we consider the logarithmic capacity $\mathrm{Cap}K$ for a compact sets $K \subset \mathbb{C}$. We note that $\mathrm{Cap}\{|z| \leq r\} = r$ and $\mathrm{Cap}(rK) = r\mathrm{Cap}K$ for $r > 0$. And we define the *capacity density* F_C of a closed set $C \subset \mathbb{C}$ by

$$F_C = \inf\left\{\frac{\mathrm{Cap}\{z \in C; |z-a| \leq r\}}{r}; a \in C, r \in (0, \mathrm{diam}C]\right\},$$

where $\mathrm{diam}C$ denotes the Euclidean diameter of C. Then we have the following estimate.

Theorem 2.11 (Pommerenke [29]).

$$\frac{{E_D}^2}{32} \leq F_{\partial D} \leq E_D.$$

Combining this theorem with the above lemmas, we have a

Corollary 2.12.

$$\frac{1}{2}\log\frac{1}{F_{\partial D}} - \frac{7}{2}\log 2 \leq M_D^\circ \leq \log\frac{1}{F_{\partial D}} + \log 3.$$

In particular, if ∂D contains ∞, we have

$$\log\frac{1}{F_{\partial D}} - \frac{5}{2}\log 2 \leq M_D^\circ \leq \log\frac{1}{F_{\partial D}} + \log 3.$$

Pommerenke originally defined uniform perfectness for a closed subset C of $\mathbb{C}$ (containing ∞), in fact, his definition is equivalent to that C is uniformly perfect iff $F_C > 0$. Of course, for each component of $\mathbb{C} \setminus C$ we have the same estimates of other domain constants by the bounds depending only on F_C.

Restricted modulus. Hereafter, we should consider only that domain D which is a hyperbolic subdomain of $\mathbb{C}$, because by the domain constants below we cannot distinguish D from $D \setminus \{\infty\}$ if D contains ∞. Beardon and Pommerenke introduced in [3] a quantity $\beta_D(z)$ which skillfully works on estimation of the hyperbolic metric of a plane domain D. The quantity β_D is defined here as the following: for $z \in D$, we set

$$\beta_D(z) = \inf\left\{2\left|\log\frac{|z-a|}{|b-a|}\right|; a, b \in \partial D, |z-a| = \delta_D(z)\right\}.$$

We should notice the reader that the above definition differs from the original one in extra factor 2.

Beardon and Pommerenke showed in [3] the next

Theorem 2.13 (Beardon-Pommerenke). *If D is a hyperbolic subdomain of $\mathbb{C}$, then we have an estimate*

$$\frac{1}{(8\sqrt{2} + \beta_D(z))\delta_D(z)} \leq \rho_D(z) \leq \frac{\pi}{2\beta_D(z)\delta_D(z)}.$$

Also, they remark in [3] that the constant $8\sqrt{2}$ can be replaced by 2×4.76 but not by 2×4.37. Further, we set

$$B_D := \sup_{z \in D} \beta_D(z),$$

which will be called the *restricted modulus* of D.

Concerning β_D and B_D, first we exhibit the following lemma, which will be proved in Section 7.

Lemma 2.14. *Let D be a hyperbolic subdomain of $\mathbb{C}$. If $\mu_D^\circ(z) = 2\log\frac{1}{r}$, we have the following best possible estimate.*

$$\max\left\{2\log\frac{1-r}{2r}, 0\right\} \le \beta_D(z) \le 2\log\frac{1}{r}. \tag{2.9}$$

We note that $\log\frac{1-r}{2r} \ge 0$ if and only if $r \le \frac{1}{3}$, in this case $\beta_D(z) \ge 2\log\frac{1}{r} + 2\log\frac{1-r}{2} \ge 2\log\frac{1}{r} + 2\log\frac{1}{3}$. On the other hand, this inequality trivially holds if $r > \frac{1}{3}$, thus we have the following

Corollary 2.15.

$$\mu_D^\circ(z) - 2\log 3 \le \beta_D(z) \le \mu_D^\circ(z).$$

Thus, immediately we obtain the following result as a corollary.

Theorem 2.16. *Let D be a hyperbolic subdomain of $\mathbb{C}$, then*

$$M_D^\circ - 2\log 3 \le B_D \le M_D^\circ.$$

Quasi-hyperbolic metric. For a proper subdomain D of $\mathbb{C}$, a continuous Riemannian metric $\frac{|dz|}{\delta_D(z)}$ is called the quasi-hyperbolic metric. If D is simply connected, the following estimate easily follows from the Koebe one-quarter theorem and Schwarz's lemma (see [28]): for each $z \in D$,

$$\frac{1}{4\delta_D(z)} \le \rho_D(z) \le \frac{1}{\delta_D(z)}.$$

For a general hyperbolic domain $D \subset \mathbb{C}$, it also holds that $\rho_D(z) \le 1/\delta_D(z)$, but $1/\delta_D(z)\rho_D(z)$ need not to be bounded. On the other hand, the quasi-hyperbolic metric is always comparable to the Hahn metric $\hat{\rho}_D = \hat{\rho}_D(z)|dz|$.

Theorem 2.17 (Minda [24]). *For any proper subdomain D of $\mathbb{C}$, we have*

$$\frac{1}{4\delta_D(z)} \le \hat{\rho}_D(z) \le \frac{1}{\delta_D(z)}.$$

Now we introduce a domain constant

$$C_D := \inf_{z\in D} \delta_D(z)\rho_D(z).$$

Then, by the theorem above, we also have $\delta_D(z)\rho_D(z) \le \frac{\rho_D(z)}{\hat{\rho}_D(z)} \le 4\delta_D(z)\rho_D(z)$, therefore we obtain the next

Corollary 2.18. *For a hyperbolic subdomain of $\mathbb{C}$, it holds that*

$$C_D \le K_D \le 4C_D.$$

On the other hand, Theorem 2.13 directly produces the following

Theorem 2.19. *If D is a hyperbolic subdomain of $\mathbb{C}$, then we have an estimate*

$$\frac{1}{8\sqrt{2}+B_D} \leq C_D \leq \frac{\pi}{2B_D}.$$

As for C_D, further Minda showed the next result.

Theorem 2.20 (Minda [25]). *Let D be a hyperbolic subdomain of $\mathbb{C}$, then we have*

$$\frac{\tanh I_D}{4} \leq C_D \leq \frac{2\sqrt{3}I_D}{\sqrt{\pi^2+4{I_D}^2}} \leq \frac{2\sqrt{3}}{\pi}I_D,$$

and moreover if D is not simply connected, we also have

$$\frac{1}{\sqrt{8N_D}} \leq C_D \leq \sqrt{\frac{6}{N_D}}.$$

In fact, Minda [25] showed that $\frac{\tanh I_D}{4} \leq C_D \leq \sqrt{\frac{6}{N_D}}$, and the other inequalities follow from this and fundamental estimates previously stated.

Finally, we present the result connecting C_D with the round modulus M_D°. A proof of this will be given in Section 7.

Theorem 2.21. *For a hyperbolic subdomain D of $\mathbb{C}$, if $M_D^\circ > 0$ we have the following estimate.*

$$\frac{1}{M_D^\circ+8\sqrt{2}} \leq C_D \leq \inf_{0<k\leq 1/2} \frac{\pi}{2}\frac{1-e^{-kM_D^\circ}}{M_D^\circ \sin k\pi} \leq \frac{\pi}{2}\frac{1-e^{-M_D^\circ/2}}{M_D^\circ} < \frac{\pi}{2M_D^\circ}. \tag{2.10}$$

In particular, tending k to 0, we also see that $C_D \leq \frac{1}{2}$, thus we have the following

Corollary 2.22. *For a hyperbolic subdomain D of $\mathbb{C}$, if $M_D^\circ > 0$ we have*

$$C_D \leq \min\left\{\frac{1}{2}, \frac{\pi}{2M_D^\circ}\right\}.$$

Remark. The assumption that $\infty \notin D$ is essential in these theorem. In fact, if let $D = \Delta^* = \{z \in \widehat{\mathbb{C}}; 1 < |z| \leq \infty\}$, we have $\rho_{\Delta^*}(z) = \frac{1}{|z|^2-1}$ and $\delta_{\Delta^*}(z) = |z|-1$, hence

$$\delta_{\Delta^*}(z)\rho_{\Delta^*}(z) = \frac{1}{|z|+1} \to 0$$

as $z \to \infty$, whereas $\tanh I_{\Delta^*} = \tanh I_\Delta = 1$ and $M_\Delta^\circ = 0$.

By summing up the results in this section, we have the following thorem, which is of course well-known.

Theorem 2.23. *The following conditions are mutually equivalent for a hyperbolic subdomain D of $\widehat{\mathbb{C}}$.*

(1) *D is modulated, i.e., $M_D < \infty$.*
(2) *D is roundly modulated, i.e., $M_D^\circ < \infty$.*
(3) *The injectivity radius I_D is positive.*
(4) *The Hahn metric is comparable to the Poincaré metric, i.e., $K_D > 0$.*
(5) *The norm N_D of the Schwarzian derivative of a universal covering is finite.*

Furthermore, if $\infty \notin D$, we can add the following conditions to the above list.

(6) *The restricted modulus B_D is finite.*
(7) *The quasi-hyperbolic metric is comparable to the hyperbolic metric, i.e., $C_D > 0$.*

3. Conformal, quasiconformal and Möbius invariance.

Now we gather some results concerning the invariance (or, variance) of the domain constants under the various mappings. First of all, constants M_R, I_R and K_R are conformally invariant by definition. In other words, if $f : R \to R'$ is a conformal (=biholomorphic) mapping, we have $M_R = M_{R'}, I_R = I_{R'}$, and $K_R = K_{R'}$. Moreover, constants M_R and I_R are quasi-invariant, i.e., the next theorem holds.

Theorem 3.1. *Suppose that there exists a K-quasiconformal homeomorphism ($K \geq 1$) of a hyperbolic Riemann surface R onto another surface R'. Then it follows that*

$$\frac{1}{K} M_R \leq M_{R'} \leq K M_R, \text{ and } \frac{1}{K} I_R \leq I_{R'} \leq K I_R.$$

The first part directly follows from the quasi-invariance of moduli of annuli. The second part, seeming to be less trivial than the first, is obtained by (2.2) and the next

Proposition 3.2 (Wolpert [35]). *The hyperbolic length of a closed geodesic is quasi-invariant. More presicely, let $f : R \to R'$ be a K-quasiconformal homeomophism and α a closed curve in R, then we have an inequality:*

$$\frac{1}{K} \ell_R[\alpha] \leq \ell_{R'}[f(\alpha)] \leq K \ell_R[\alpha].$$

For convenience of the reader, we shall give a proof of this proposition in Section 4. In particular, we know that the modulatedness of a hyperbolic Riemann surface is quasiconformally invariant. On the other hand, constants N_D, M_D°, B_D and C_D for a plane domain D are not conformally invariant, but some kind of estimate can be deduced. As for the constant N_D, the following absolute estimate is obtained.

Theorem 3.3. *If hyperbolic domains D and D' are conformally equivalent, then we have*

$$|N_D - N_{D'}| \leq 12.$$

Proof. Let $f : D \to D'$ be a conformal map and $p : \mathbb{H} \to D$ a holomorphic universal cover of D. Then $f \circ p$ is a holomorphic universal cover of D' and it holds that $S_{f\circ p} = p^*(S_f) + S_p$, where p^* denotes pull-back of quadratic differentials by p, i.e., $p^*\varphi = (\varphi \circ p)(p')^2$. Since $\|S_f\|_D := \sup_{z\in D} |S_f(z)|\rho_D(z)^{-2} \leq 12$ by the theorem of Beardon-Gehring [2], we have

$$\begin{aligned} |N_D - N_{D'}| &\leq \sup_{z\in\mathbb{H}} |p^* S_f(z)|\rho_{\mathbb{H}}(z)^{-2} \\ &= \sup_{z\in\mathbb{H}} |S_f(p(z))||p'(z)|^2\rho_{\mathbb{H}}(z)^{-2} \\ &= \sup_{z\in\mathbb{H}} |S_f(p(z))|\rho_D(p(z))^{-2} \leq 12, \end{aligned}$$

where we used a relation $\rho_{\mathbb{H}}(z) = \rho_D(p(z))|p'(z)|$. □

The constant N_D is Möbius invariant for $S_p = S_{L\circ p}$ for any Möbius transformation L. Other constants M_D°, B_D and C_D are not even Möbius invariant, but we obtain several estimates by quite elementary calculations.

Theorem 3.4. *Let D be a hyperbolic subdomain of $\widehat{\mathbb{C}}$. For any $L \in$ Möb, we have*

$$\frac{1}{2}M_D^\circ - \log\frac{4}{3} \leq M_{L(D)}^\circ. \tag{3.1}$$

Further, in case $L(D) \subset \mathbb{C}$, we have a better estimate:

$$M_D^\circ - \log 2 \leq M_{L(D)}^\circ.$$

Remark. As we shall see in the proof (in Section 7), the constant $\frac{1}{2}$ in (3.1) is best possible in the following sense. Suppose that there exist constants $c > 0$ and $d \in \mathbb{R}$ such that

$$cM_D^\circ + d \leq M_{L(D)}$$

for any hyperbolic subdomain D of $\widehat{\mathbb{C}}$ and $L \in$ Möb, then c must not be greater than $\frac{1}{2}$.

As to the constant C_D, we have the next result as an immediate consequence of Corollary 2.18. Compare the result in [25].

Proposition 3.5. *Let D and D' be conformally equivalent hyperbolic domains in $\mathbb{C}$, then it holds that*

$$C_D \leq 4C_{D'}.$$

Proof. Since K_D is conformally invariant, Corollary 2.18 yields that

$$C_D \leq K_D = K_{D'} \leq 4C_{D'}.$$

□

A quasiconformal variance of the other constants is obtained, at least in principle, by combining Theorem 3.1 with the comparison theorems in Section 2. For example, now we consider the contant N_D. Suppose that $f : D \to D'$ is a K-quasiconformal homeomorphism. Noting that $\coth x \leq 1+\frac{1}{x}$ for $x > 0$, we have the following estimate by Minda's theorem:

$$\begin{aligned} N_{D'} &\leq 6 \coth^2 I_D \leq 6 \left(1 + \frac{1}{I_D}\right)^2 \leq 6 \left(1 + \frac{K}{I_{D'}}\right)^2 \\ &\leq 6 \left(1 + \frac{K}{\pi}\sqrt{2(N_D - 2)}\right)^2 \\ &\leq 12 + \frac{24K^2}{\pi^2}(N_D - 2). \end{aligned}$$

Of course, a more direct method may give a better estimate, which the author does not know unfortunately.

4. Geometry of annuli.

Here, We state standard facts about geometry of annuli with some proofs. Let $A = A_r$ be the round annuli $\{z \in \mathbb{C}; r < |z| < 1\}$ $(0 < r < 1)$. A holomorphic universal cover $p : \mathbb{H} \to A$ is concretely given by

$$p(z) = \exp\left(\frac{i \log 1/r}{\pi} \log z\right), \quad (z \in \mathbb{H}), \tag{4.1}$$

where we take the principal branch of $\log z$, that is, $\operatorname{Im} \log z = \arg z \in (0, \pi)$. As is easily observed, the covering transformation group of $p : \mathbb{H} \to A$ is an infinite cyclic group generated by $g : z \mapsto e^{\ell} z$, where

$$\ell = \frac{2\pi^2}{\log 1/r} > 0.$$

Therefore, the image of the axis $\operatorname{ax}(g) = \{iy; y > 0\}$ under p is the core curve $\operatorname{Core} A = \{z; |z| = \sqrt{r}\}$. In particular, we can calculate the hyperbolic length of the core curve of A as

$$\ell(\operatorname{Core} A) = \int_1^{e^{\ell}} \frac{dy}{2y} = \frac{\ell}{2} = \frac{\pi^2}{\log 1/r} = \frac{\pi^2}{m(A)}, \tag{4.2}$$

thus we conclude by (2.2) that

$$I_A = \frac{\pi^2}{2m(A)}. \tag{4.3}$$

On the other hand, since the Jenkins-Strebel differential for Core$A$ is a constant multiple of dz^2/z^2, we can easily deduce that

$$E(\mathrm{Core}A) = \frac{2\pi}{m(A)}.$$

Here, we prepare an elementary lemma for later use.

Lemma 4.1. *Suppose g is a hyperbolic transformation of $\mathbb{H}$ defined by $g(z) = e^{\ell}z$ with $\ell > 0$.*

Let $\theta_1, \theta_2 \in (0, \pi)$ with $\theta_1 < \theta_2$, then $\mathbb{H}(\theta_1, \theta_2)/\langle g \rangle$ is biholomorphic to an annulus with modulus $\frac{2\pi}{\ell}(\theta_2 - \theta_1)$, where $\mathbb{H}(\theta_1, \theta_2)$ denotes the set $\{z \in \mathbb{H}; \theta_1 < \arg z < \theta_2\}$.

Proof. Choose an $r \in (0,1)$ so that $\ell = \frac{2\pi^2}{\log 1/r}$ and let $p : \mathbb{H} \to A_r$ be as in (4.1). Then, it is clear that $\mathbb{H}(\theta_1, \theta_2)/\langle g \rangle$ is biholomorphically equivalent to $p(\mathbb{H}(\theta_1, \theta_2)) = \{z; \frac{\theta_1}{\pi}\log\frac{1}{r} < \log\frac{1}{|z|} < \frac{\theta_2}{\pi}\log\frac{1}{r}\}$, which is an annulus with modulus $\frac{\theta_2-\theta_1}{\pi}\log\frac{1}{r} = \frac{2\pi}{\ell}(\theta_2 - \theta_1)$. □

As an application, we give here a proof of Proposition 3.2 (due to Wolpert [35]).

Proof of Proposition 3.2. Let $f : R \to R'$ be a K-quasiconformal homeomorphism and α a closed curve in R. We consider a holomorphic universal cover $p : \mathbb{H} \to R$ and $p' : \mathbb{H} \to R'$ with covering transformation groups Γ and Γ', respectively; and let $\tilde{f}$ be a lift of f, i.e., $\tilde{f} : \mathbb{H} \to \mathbb{H}$ is a K-quasiconformal homeomorphism satisfying that $p' \circ \tilde{f} = f \circ p$. Suppose that $\gamma \in \Gamma$ covers α, and let $\gamma' = \tilde{f} \circ \gamma \circ \tilde{f}^{-1} \in \Gamma'$. Then γ' covers $\alpha' := f(\alpha)$. Now we consider annuli $A = \mathbb{H}/\langle\gamma\rangle$ and $A' = \mathbb{H}/\langle\gamma'\rangle$. Then $\tilde{f}$ naturally induces a K-quasiconformal homeomorphism $\hat{f} : A \to A'$. The quasi-invariance of moduli of annuli implies that

$$\frac{1}{K}m(A) \le m(A') \le Km(A).$$

On the other hand, $m(A) = \frac{\pi^2}{l_\gamma} = \frac{\pi^2}{\ell_R[\alpha]}$ and so on, hence we obtain that

$$\frac{1}{K}\ell_R[\alpha] \le \ell_{R'}[\alpha'] \le K\ell_R[\alpha].$$

□

Next, we give a concrete form of the hyperbolic metric of A. For each $z \in A$, we take a point $\zeta \in \mathbb{H}$ so that $p(\zeta) = z$. Then we have $\rho_A(z)|p'(\zeta)| = p^*\rho_A(\zeta) = \rho_{\mathbb{H}}(\zeta) = \frac{1}{2\mathrm{Im}\zeta}$. Letting $\theta = \arg\zeta \in (0, \pi)$, since $|z| = |p(\zeta)| = \exp(-\frac{\theta}{\pi}\log\frac{1}{r})$, we have $\theta = \pi\frac{\log 1/|z|}{\log 1/r}$.

Therefore,

$$\frac{1}{\rho_A(z)} = 2\mathrm{Im}\zeta \cdot |p'(\zeta)| = 2\mathrm{Im}\zeta \cdot \frac{\log 1/r}{\pi|\zeta|}|z| = \frac{2}{\pi}\log\frac{1}{r}\cdot|z|\sin\theta$$
$$= \frac{2}{\pi}\log\frac{1}{r}\cdot\sin\left(\pi\frac{\log 1/|z|}{\log 1/r}\right).$$

Now we obtain that

$$\rho_A(z) = \frac{\pi}{2|z|\log 1/r\sin\left(\pi\frac{\log 1/|z|}{\log 1/r}\right)}. \tag{4.4}$$

We should remark that (4.4) remains valid if r tends to 0, i.e.,

$$\rho_{A_0}(z) = \frac{1}{2|z|\log 1/|z|}.$$

Further, we note that for a general round annulus $A = \{z; r_1 < |z-a| < r_2\}$, it follows that

$$\rho_A(z) = \frac{\pi}{2|z-a|m(A)\sin\left(\pi\frac{\log r_2/|z-a|}{\log r_2/r_1}\right)}. \tag{4.5}$$

Using this explicit form of the hyperbolic metric of a round annulus, we determine the constant N_{A_r}. A direct computation shows that the Schwarzian derivative of $p : \mathbb{H} \to A_r$ has a simple form:

$$S_p(z) = \frac{1}{2}\left(\frac{(\log 1/r)^2}{\pi^2}+1\right)\frac{1}{z^2}.$$

Therefore,

$$N_{A_r} = \|S_p\|_{\mathbb{H}} = \sup_{z\in\mathbb{H}}\frac{1}{2}\left(\frac{(\log 1/r)^2}{\pi^2}+1\right)\frac{1}{|z|^2}(2\mathrm{Im}z)^2$$
$$= 2\left(1+\frac{(\log 1/r)^2}{\pi^2}\right) = 2+\frac{2\pi^2}{I_{A_r}^2},$$

which is the case that the equality occurs in the left hand side of (2.8).

5. Hyperbolic vs. extremal lentgth.

In this section, we shall give several estimates between the hyperbolic and extremal lengths, from which we can derive some comparisons between the constants I_R and M_R. In our proof, the collar lemma plays a significant role. We say that a simple closed geodesic α in R has a *collar of width* $\omega > 0$ around it if the $\omega/2$-neighborhood $\{q \in R; \mathrm{dist}_R(q,\alpha) < \omega/2\}$ of α is homeomorphic to an annulus, where $\mathrm{dist}_R(q,\alpha)$ is the hyperbolic distance of q from α. The following form of the collar lemma fits our

present aim, for which the reader may consult the book [4]. See also [12] for a short proof.

Theorem 5.1 (The collar lemma). *A simple closed geodesic α of length $\lambda > 0$ in a hyperbolic Riemann surface R has at least a collar of width ω around it, where ω is the positive number determined by the relation*

$$\sinh\omega \sinh\lambda = 1. \tag{5.1}$$

Further, this ω is best possible.

By the aid of the collar lemma, now we can show the following

Theorem 5.2 (Comparison theorem). *For any $[\alpha] \in \mathcal{S}_R$, we have the following estimate:*

$$\frac{2}{\pi}\ell[\alpha] \leq E[\alpha] \leq \frac{\ell[\alpha]}{\arctan\left(\frac{1}{\sinh \ell[\alpha]}\right)}. \tag{5.2}$$

Further, $\frac{2}{\pi}\ell[\alpha] = E[\alpha]$ for some $[\alpha] \in \mathcal{S}_R$ with $\ell[\alpha] > 0$ if and only if R is an annulus with $m(R) < \infty$.

By elementary, but slightly boring calculations, we can see that the function $e^x \arctan(\frac{1}{\sinh x})$ of $x > 0$ monotonically increases from $\frac{\pi}{2} = 1.57\cdots$ to 2. Therefore, we have a

Corollary 5.3 (A simpler form). *For any $[\alpha] \in \mathcal{S}_R$, we have a slightly weak but simpler estimate:*

$$\frac{2}{\pi}\ell[\alpha] \leq E[\alpha] \leq \frac{2}{\pi}\ell[\alpha]e^{\ell[\alpha]} \tag{5.3}$$

It seems that even Corollary 5.3 is rather sharp at least when $\ell[\alpha]$ is sufficiently small. Maskit [21] has obtained a similar result:

$$\frac{2}{\pi}\ell[\alpha] \leq E[\alpha] \leq \frac{\ell[\alpha]}{\arcsin\left(e^{-\ell[\alpha]}\right)}$$

(in our present notations), from which he derived also that $\ell[\alpha]/E[\alpha] \to \frac{\pi}{2}$ as $\ell[\alpha] \to 0$ and that $E[\alpha] \leq \ell[\alpha]e^{\ell[\alpha]}$. Further we remark that it holds that $\arcsin(e^{-x}) < \arctan(\frac{1}{\sinh x})$ for any $x > 0$, so Theorem 5.2 is an improvement of Maskit's result.

Proof of Theorem 5.2. Let α be a simple closed curve in R and $p : \mathbb{H} \to R$ be a holomorphic universal cover of R. And let $\gamma \in \Gamma$ be a covering transformation which covers α, i.e., there exists a lift $\tilde{\alpha} : [0,1] \to \mathbb{H}$ of α such that $\gamma(\tilde{\alpha}(0)) = \tilde{\alpha}(1)$. First, we shall show the right hand side inequality. If γ is parabolic (i.e., $\ell[\alpha] = 0$), it holds that $E[\alpha] = 0$ because the punctured disk (so, of infinite modulus) can be conformally embedded so that an embedded loop around the puncture is freely homotopic to α. Thus, we may assume that γ is a hyperbolic transformation, in other words, γ has a

form $\gamma(z) = e^{2\lambda}z$, where $\lambda > 0$ is the hyperbolic length of $[\alpha]$ and that α is a simple closed geodesic. By the collar lemma, α has a collar A of width ω satisfying (5.1). Let $\widetilde{A}$ be the connected component of $p^{-1}(A)$ which contains the axis $\mathrm{ax}(\gamma)$ of γ, i.e., the imaginary axis. Since $p(\mathrm{ax}(\gamma)) = \alpha$, we see that

$$\begin{aligned}\widetilde{A} &= \{z \in \mathbb{H}; \mathrm{dist}_{\mathbb{H}}(z, \mathrm{ax}(\gamma)) < \frac{\omega}{2}\} \\ &= \{z \in \mathbb{H}; |\arg z - \frac{\pi}{2}| < \theta\},\end{aligned}$$

where $\theta > 0$ is the angle satisfying $\tan\frac{\theta}{2} = \tanh\frac{\omega}{2}$, or equivalently,

$$\tan\theta = \sinh\omega. \tag{5.4}$$

Thus, Lemma 5.4 yields that $m(A) = \frac{2\pi\theta}{\lambda}$. Now, we conclude by Theorem 2.1 that

$$\frac{2\pi}{E[\alpha]} \geq m(A) = \frac{2\pi\theta}{\lambda} = \frac{2\pi}{\lambda}\arctan\left(\frac{1}{\sinh\lambda}\right),$$

which is equivalent to the right hand side inequality of (5.2).

Next, we show the left hand side inequality: $\frac{2}{\pi}\ell[\alpha] \leq E[\alpha]$. However this is a known result, we shall give a different proof from Maskit's one [21] (using the length-area method) and from Matsuzaki's one [22] (using the Poincaré theta series). Let $R_0 \in \mathcal{A}_R$ with $\alpha_0 = \mathrm{Core}R_0 \in [\alpha]$ and $m(A) < \infty$. Then, since $\rho_R \leq \rho_{R_0}$ on R_0, we have

$$\ell[\alpha] \leq \ell(\alpha_0) = \int_{\alpha_0}\rho_R(z)|dz| \leq \int_{\alpha_0}\rho_{R_0}(z)|dz| = \frac{\pi^2}{m(A)}$$

from (4.2). By virtue of the Jenkins-Strebel theorem (Theorem 2.1), we have $\ell[\alpha] \leq \frac{\pi}{2}E[\alpha]$. From the above proof, we can also observe that the equality occurs only when $\ell[\alpha] = 0$ or $R = R_0$, i.e., R is an annulus itself. □

Remark. We can prove that $\ell[\alpha] \leq \frac{\pi}{2}E[\alpha]$ also using the annular covering method. Let $h : \mathbb{H} \to A := \mathbb{H}/\langle\gamma\rangle$ be the natural projection. Note that A is an annulus with modulus $m(A) = \frac{\pi^2}{l_\gamma} = \frac{\pi^2}{\ell[\alpha]}$ (see Section 4). The universal cover $p : \mathbb{H} \to R$ induces the annular cover $q : A \to R$ with respect to α such that $p = q \circ h$. By construction of q, any other lift of α via q is *not* closed than $\hat{\alpha} = h(\tilde{\alpha})$, where $\tilde{\alpha} : [0,1] \to \mathbb{H}$ is a lift of α via p such that $\gamma(\tilde{\alpha}(0)) = \tilde{\alpha}(1)$. Let R_0 be the characteristic ring domain for α (see Theorem 2.1, and A_0 be the unique doubly connected component of $q^{-1}(R_0)$. Here we should note that $q : A_0 \to R_0$ is biholomorphic. Since A_0 is contained essentially in A, by the monotonicity of the moduli of annuli, we have

$$\frac{\pi^2}{\ell[\alpha]} = m(A) \geq m(A_0) = m(R_0) = \frac{2\pi}{E[\alpha]}.$$

As Maskit remarked in [21], if $[\alpha] \in \mathcal{S}_R$ with $\ell[\alpha] > 0$ is a boundary curve (i.e., α divides R into two parts, one of which is an annulus), then we can take a collar $A = p(\tilde{A})$ around α in the above proof, where

$$\tilde{A} = \{z \in \mathbb{H}; 0 < \arg z < \frac{\pi}{2} + \theta\},$$

and θ satisfies (5.4) for a suitable choice of $p : \mathbb{H} \to R$. Thus,

$$\begin{aligned} \frac{2\pi}{E[\alpha]} \geq m(A) = \frac{\pi(\theta + \pi/2)}{\lambda} &= \frac{\pi}{\lambda}\left(\frac{\pi}{2} + \arctan\left(\frac{1}{\sinh\lambda}\right)\right) \\ &\geq \frac{\pi}{\lambda}\left(\frac{\pi}{2} + \frac{\pi}{2}e^{-\lambda}\right), \end{aligned}$$

and we also have the following

Theorem 5.4. *For any boundary simple closed curve α in R with $\ell[\alpha] > 0$, we have a stronger estimate:*

$$\frac{2}{\pi}\ell[\alpha] \leq E[\alpha] \leq \frac{4}{\pi}\frac{\ell[\alpha]}{1 + e^{-\ell[\alpha]}} < \frac{4}{\pi}\ell[\alpha].$$

Since any $[\alpha] \in \mathcal{S}_R$ is a boundary curve for a triply connected domain R, we obtain the next

Corollary 5.5. *For any triply connected planar Riemann surface R with $I_R > 0$, we have a better estimate:*

$$2I_R < \frac{\pi^2}{M_R} < 4I_R.$$

6. An application to quadratic differentials.

In this section, we consider complex Banach spaces of integrable and (hyperbolically) bounded holomorphic quadratic differentials on a hyperbolic Riemann surface. Presicely, let $A_2(R)$ and $B_2(R)$ be the complex Banach spaces consisting of all holomorphic quadratic differentials $\varphi = \varphi(z)dz^2$ on a hyperbolic Riemann surface R with finite norms $\|\varphi\|_1 = \iint_R |\varphi| = \iint_R |\varphi(z)|dxdy$ and $\|\varphi\|_\infty = \sup_{q\in R} |\varphi(q)|\rho_R(q)^{-2}$, respectively. A neccesary and sufficient condition for $A_2(R)$ being (continuously) contained in $B_2(R)$ is known to be that

$$\inf_{\gamma\in\Gamma:\ \text{hyperbolic}} |\mathrm{tr}\gamma| > 2,$$

where Γ is a Fuchsian group such that $R \cong \mathbb{H}/\Gamma$. In particular, we sholud have $A_2(R) \subset B_2(R)$ for any modulated Riemann surface. As an application of our arguments in previous sections, we shall give a simple proof for this fact with a concrete estimate for the quantity

$$\kappa_R := \sup\{\|\varphi\|_\infty; \varphi \in A_2(R) \text{ with } \|\varphi\|_1 = 1\}.$$

Here we note that $\kappa_R < \infty$ if and only if $A_2(R) \subset B_2(R)$. The following is due to Matsuzaki [22].

Theorem 6.1. *For each $[\alpha] \in \mathcal{S}_R$, we have $E[\alpha] \le \kappa_R \ell[\alpha]^2$.*

Proof. We may assume that $E[\alpha] > 0$. Let φ_0 be the Jenkins-Strebel differential for α as in Theorem 2.1 with $\|\varphi_0\|_1 = 1$. Then, for $\alpha' \in [\alpha]$,

$$\begin{aligned} E[\alpha]^{1/2} &\le \int_{\alpha'} |\varphi_0|^{1/2}|dz| = \int_{\alpha'} |\varphi_0 {\rho_R}^{-2}|^{1/2}\rho_R|dz| \\ &\le \|\varphi_0\|_\infty^{1/2} \int_{\alpha'} \rho_R|dz|. \end{aligned}$$

Since $\alpha' \in [\alpha]$ is arbitrary, we conclude that $E[\alpha] \le \|\varphi_0\|_\infty \ell[\alpha]^2$, which proves the theorem. □

On the other hand, by the mean value property of holomorphic functions, we can easily estimate the hyperbolic sup norm $\|\varphi\|_\infty$ by L^1-norm $\|\varphi\|_1$ for a modulated surface as in the following way.

Proposition 6.2. *For any hyperbolic Riemann surface R, it holds that*

$$\kappa_R \le \frac{1}{\pi} \coth^2 I_R,$$

which implies that $A_2(R) \subset B_2(R)$ for a modulated surface R.

Proof. Fix an arbitrary point $q \in R$. Let $p : \Delta = \{z \in \mathbb{C}; |z| < 1\} \to R$ be a holomorphic universal cover with $p(0) = q$. We denote by $\tilde{\varphi}$ the pull-back of $\varphi \in A_2(R)$ by p, i.e., $\tilde{\varphi}(z)dz^2 = \varphi(p(z))p'(z)^2dz^2$. Then, we can regard $\tilde{\varphi}$ as a holomorphic function satisfying the functional equations: $\tilde{\varphi}(\gamma(z))\gamma'(z)^2 = \tilde{\varphi}(z)$ for all covering transformation γ. Here, note that $|\varphi\rho_R^{-2}|(q) = |\tilde{\varphi}(0)|$ since $\rho_\Delta(0) = 1$. By the mean value property, for any $r \in (0,1)$ we have

$$\tilde{\varphi}(0) = \frac{1}{\pi r^2} \iint_{|z|<r} \tilde{\varphi}(z)dxdy.$$

Since p is injective in the disk $\{z; |z| < \tanh \iota_R(q)\} = D_\Delta(0, \iota_R(q))$, for $r = \tanh \iota_R(q)$ we have

$$\begin{aligned} |\tilde{\varphi}(0)| &\le \frac{1}{\pi r^2} \iint_{|z|<r} |\tilde{\varphi}(z)|dxdy = \frac{1}{\pi r^2} \iint_{D_R(q,\iota_R(q))} |\varphi| \\ &\le \frac{1}{\pi r^2}\|\varphi\|_1 = \frac{1}{\pi} \coth^2 \iota_R(q) \cdot \|\varphi\|_1. \end{aligned}$$

Thus we have the assertion that $\|\varphi\|_\infty \le \frac{1}{\pi} \coth^2 I_R \cdot \|\varphi\|_1$. □

Remarks. By the result above, we see that $\kappa_R = O({I_R}^{-2})$ as $I_R \to 0$. In fact, Matsuzaki [22] showed that $\kappa_R = O({I_R}^{-1})$ as $I_R \to 0$ by an argument using the Marden-Margulis constant. (Further, he gave a comparison theorem between κ_R and other geometric quantities.) Following Matsuzaki's method (in torsion-free case), we can similarly show that

$$\kappa_R \leq \frac{2}{\pi}\left(\frac{\mu_0}{I_R} + 1\right),$$

where $\mu_0 = \sinh^{-1}(1) = \log(1+\sqrt{2}) = 0.8813\cdots$ is the number which appered in [36], but this estimate is not so sharp when I_R is not small.

By combining Theorem 6.1 with Proposition 6.2, we obtain the following

Corollary 6.3. *For each* $[\alpha] \in \mathcal{S}_R$, *we have*

$$E[\alpha] \leq \frac{1}{\pi}\coth^2 I_R \cdot \ell[\alpha]^2.$$

Since $I_R = \inf \frac{\ell[\alpha]}{2}$ and $\frac{2\pi}{M_R} = \inf E[\alpha]$, we see that

$$\frac{2\pi}{M_R} \leq \frac{4}{\pi}{I_R}^2 \cdot \coth^2 I_R,$$

thus (2.6) is now proved.

7. Elementary calculations.

In this section, we shall give proofs for other theorems by quite elementary but rather tedious calculations.

Proof of Lemma 2.14. The inequality $\beta_D(z) \leq \mu^\circ_D(z)$ is implicitly proved in [3]. For reader's convenience, we include a proof of this. We write simply $\beta = \beta_D(z), \mu = \mu^\circ_D(z)$ and $\delta = \delta_D(z)$. And we may assume that $\beta > 0$. Take $a \in \partial D$ such that $|z-a| = \delta$ and let A be the annulus

$$A = \{w \in \mathbb{C}; \delta e^{-\beta/2} < |w-a| < \delta e^{\beta/2}\}.$$

For any $w \in A$, we have $\left|\log\left|\frac{z-a}{w-a}\right|\right| < \frac{\beta}{2}$, which implies that $A \cap \partial D = \emptyset$ by the definition of β. Since $z \in A$, we conclude that $A \subset D$. As z is separated from ∞ by A, we have $A \in \mathcal{A}^\circ_D$. Since $z \in \mathrm{Core}A$ by construction, we obtain that $\mu^\circ_D(z) \geq m(A) = \beta$, thus the first part is proved.

Next we show that $\beta \geq \max\{2\log\frac{1-r}{2r}, 0\}$ if $\mu = 2\log 1/r$. It suffices to prove in the case that $\log\frac{1-r}{2r} > 0$, that is, $0 \leq r < \frac{1}{3}$. We choose $a, b \in \partial D$ so that $\delta = |z-a|$ and $\beta = 2|\log|\frac{z-a}{b-a}||$. Let $A \in \mathcal{A}^\circ_D$ with $z \in \mathrm{Core}A$. It is sufficient to show that $\beta \geq 2\log\frac{1-r'}{2r'}$, where $m(A) = 2\log 1/r'$. Without loss of generality, we may assume that $z = 1$ and $A = \{w; r' < |w| < \frac{1}{r'}\}$ $\quad (r \leq r' < 1/3)$.

First, note that $1+r' < \frac{1}{r'} - 1$ since $r' < \frac{1}{3}$. Thus, a should satisfy that $|a| \leq r'$. In fact, if $|a| \geq \frac{1}{r'}$, then there exists a point $b \in \partial D$ such that $|b| \leq r'$ since A divides ∂D into (at least) two parts. Therefore, $|1-b| \leq 1+r' < \frac{1}{r'} - 1 \leq |1-a| = \delta_D(1)$, this is contrary to the definition of $\delta_D(1)$.

If $|b| \geq \frac{1}{r'}$, we have

$$\left|\frac{1-a}{b-a}\right| \geq \frac{1-r'}{r'+1/r'} \geq \frac{1/r'-r'}{1+r'} = \frac{1-r'}{r'} > \frac{1-r'}{2r'}.$$

If $|b| \leq r'$, we have $|\frac{1-a}{b-a}| \geq \frac{1-r'}{2r'}$, too. In both cases, we have $|\frac{1-a}{b-a}| \geq \frac{1-r'}{2r'}$, i.e.,

$$\log\left|\frac{1-a}{b-a}\right| \geq \log\frac{1-r'}{2r'} (> 0),$$

this leads to the desired inequality because $r' \geq r$ is arbitrary. As one can easily show, if $D = \{z \in \mathbb{C}; |z| > r\} \quad (0 < r < 1)$, then $\mu_D^\circ(1) = 2\log\frac{1}{r}$ and $\beta_D(1) = \max\{\log\frac{1-2r}{2r}, 0\}$, thus the above estimate is best possible. □

Proof of Theorem 3.4. In order to prove Theorem 3.4, we may assume that D is a round annulus. First, we note that a Möbius transformation L which is not a similarity is always written as $L(z) = \frac{c}{z-a} + d$. Since the quantity M_D° is invariant under similarities, it is sufficient to prove the theorem in the case that $D = A_r = \{z; r < |z| < 1\} \quad (0 < r < 1)$ and $L_a(z) = \frac{1}{z-a} \quad (a \in [0, \infty))$. Further note that $L_a(a > 0)$ can be decomposed to the form

$$L_a = S \circ L_{r/a} \circ T,$$

where S is a similarity: $S(z) = -a^2 z/r - a/r$ and $T(z) = r/z$. Since $T(A_r) = A_r$, we obtain that

$$M^\circ_{L_a(A_r)} = M^\circ_{L_{r/a}(A_r)}.$$

Thus, it is sufficient to prove the following

Claim. For $A = \{z; r < |z| < 1\} \quad (0 < r < 1)$ and $L(z) = \frac{1}{z-a}$, where $0 \leq a \leq \sqrt{r}$, it holds that

$$M^\circ_{L(A)} \geq \frac{1}{2} M_D^\circ + \log\frac{3}{4} = \frac{1}{2}\log\frac{1}{r} + \log\frac{3}{4}.$$

Moreover, if $L(A) \subset \mathbb{C}$, then

$$M^\circ_{L(A)} > M_D^\circ + \log\frac{1}{2} = \log\frac{1}{r} + \log\frac{1}{2}.$$

Now we prove Claim. When $a \neq r$, we observe that $L(\pm 1) = \frac{1}{\pm 1 - a} \quad (L(\pm r) = \frac{1}{\pm r - a})$ are the end points of a diameter of the circle $C_0 = L(\{|z| = 1\})$ (resp. $C_1 = L(\{|z| = r\})$). Thus $\frac{a}{1-a^2}$ and $\frac{a}{r^2-a^2}$ are the centers of C_0 and C_1, respectively. Further, note that under the above hypothesis, $L(A) \subset \mathbb{C}$ if and only if $0 \leq a \leq r$.

Case 1: $0 \leq a \leq r$.

Since we can treat $a = r$ as the limiting case of $a < r$, we may assume that $0 \le a < r$. In this case, the circle C_0 is contained in the inside of C_1. Since $(\frac{1}{r-a} - \frac{1}{1-a}) - (\frac{1}{-1-a} - \frac{1}{-r-a}) > 0$, we can see that

$$M^\circ_{L(A)} = \log \frac{\frac{a}{1-a^2} - \frac{1}{-r-a}}{\frac{a}{1-a^2} - \frac{1}{-1-a}} = \log \frac{1+ar}{a+r},$$

so we have

$$M^\circ_{L(A)} - \log \frac{1}{r} = \log \frac{r(1+ar)}{a+r} \ge \log \frac{r(1+r^2)}{2r} = \log \frac{1+r^2}{2} > \log \frac{1}{2},$$

which proves the second part of Claim.

Case 2: $r < a \le \sqrt{r}$.

In this case, the circle C_1 is contained in the outside of C_1. Since

$$\left(\frac{1}{1-a} - \frac{1}{-1-a}\right) - \left(\frac{1}{-r-a} - \frac{1}{r-a}\right) = 2\frac{(a^2-r)(1+r)}{(a^2-r^2)(1-a^2)} \le 0,$$

the radius of C_1 is greater than or equal to that of C_0. Therefore, we are convinced that

$$M^\circ_{L(A)} = \log \frac{\frac{a}{1-a^2} - \frac{1}{-r-a}}{\frac{a}{1-a^2} - \frac{1}{-1-a}} = \log \frac{1+ar}{a+r},$$

thus

$$\begin{aligned} M^\circ_{L(A)} - \frac{1}{2}\log \frac{1}{r} &= \log \frac{\sqrt{r}(1+ar)}{a+r} \ge \log \frac{\sqrt{r}(1+r\sqrt{r})}{\sqrt{r}+r} \\ &= \log \left\{(\sqrt{r} - \frac{1}{2})^2 + \frac{3}{4}\right\} \ge \log \frac{3}{4}, \end{aligned}$$

which proves the first part of Claim. Note that the equality holds if $r = \frac{1}{4}$ and $a = \frac{1}{2}$. Moreover, we remark that for any $c \in (0, \frac{1}{2})$,

$$\begin{aligned} M^\circ_{L(A)} - c\log \frac{1}{r} &\ge \log r^c \frac{1+r\sqrt{r}}{\sqrt{r}+r} = \log r^{c-1/2} + \log \frac{1+r\sqrt{r}}{1+\sqrt{r}} \\ &= \left(\frac{1}{2} - c\right)\log \frac{1}{r} + O(1) \quad \text{as } r \to +0, \end{aligned}$$

so the factor $\frac{1}{2}$ is best possible. □

Proof of Theorem 2.21. The inequality $1/C_D \le M^\circ_D + 8\sqrt{2}$ directly follows from Theorem 2.13 and Theorem 2.16. Now we prove the another inequality. For this, we fix any number $k \in (0, 1/2]$. Let $A = \{z \in \mathbb{C}; r_1 < |z - c_0| < r_2\} \in \mathcal{A}^\circ_D$ and $z_0 \in A$ so that $|z_0 - c_0| = (\frac{r_2}{r_1})^k r_1 = r_1^{1-k} r_2^k$, then by (4.5) we see that

$$\rho_A(z_0) = \frac{\pi}{2|z_0 - c_0| m(A) \sin \pi k}.$$

Here, we may assume that A is maximal in the class $\{A' \in \mathcal{A}_D^\circ; \mathrm{Cent}A' = c_0\}$, thus we can select z_0 so that $\delta_D(z_0) = |z_0 - c_0| - r_1$. Hence,

$$C_D \leq \delta_D(z_0)\rho_D(z_0) \leq \delta_D(z_0)\rho_A(z_0) = \frac{\pi}{2m(A)} \frac{1 - \left(\frac{r_2}{r_1}\right)^{-k}}{\sin \pi k} = \frac{\pi(1 - e^{-km(A)})}{2m(A) \sin \pi k},$$

and tending $m(A)$ to M_D° yields that

$$C_D \leq \frac{\pi(1 - e^{-kM_D^\circ})}{2M_D^\circ \sin \pi k}$$

for any $k \in (0, 1/2]$, thus the proof is now completed. □

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