On Mordell-Weil Lattices of Higher Genus Fibrations on Rational Surfaces

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Abstract

We will give an upper bound of Mordell-Weil rank r for relatively minimal fibrations of curves of genus $g \geq 1$ on rational surfaces. Under the assumption that a fibration is not locally trivial, we have $r \leq 4g+4$. Moreover the maximal case ($r = 4g + 4$) will be studied in detail. We determine the structure of such fibrations and also the structure of their Mordell-Weil lattices introduced by Shioda.

0 Introduction

(0.1) Let $f : X \longrightarrow C$ be a relatively minimal fibration of curves of genus $g \geq 1$ over a smooth projective curve C defined over an algebraically closed field k of characteristic zero, and let K be the rational function field of C. We assume that there exists a section O of f. For such a fibration, we can define the Mordell-Weil group to be the group of the K-rational points of the Jacobian J_Γ of the generic fiber Γ/K of f. Under the suitable condition, the Mordell-Weil group $J_\Gamma(K)$ is a finitely generated abelian group, so we define the Mordell-Weil rank r to be the rank of its free part. In this note we first prove the following theorem which gives an upper bound of the Mordell-Weil rank r for fibrations of genus g on rational surfaces X.

Theorem A. *(cf. Theorem 2.8) Let X be a smooth rational surface with a relatively minimal fibration $f : X \longrightarrow \mathbf{P}^1$ of curves of genus $g \geq 1$. Then we have*

$$r = \mathrm{rank} J_\Gamma(K) \leq 4g + 4.$$

Moreover we have the equality $r = 4g+4$ *if and only if* $f : X \longrightarrow \mathbf{P}^1$ *is a hyperelliptic fibration with* $K^2_{X/\mathbf{P}^1} = 4g-4$ *such that all fibers of* f *are irreducible. Here* $K_{X/\mathbf{P}^1} = K_X \otimes f^*(K^{-1}_{\mathbf{P}^1})$ *denotes the relative canonical bundle of* f.

(0.2)

If $f : X \longrightarrow \mathbf{P}^1$ is a relatively minimal rational elliptic surface with a section, it can be obtained as the minimal resolution of its Weierstrass model, and It is easy to see that all fibers of f are irreducible if and only if its Weierstrass model is smooth. Moreover we can easily construct a smooth Weierstrass fibration $f : X \longrightarrow \mathbf{P}^1$ such that X is a rational surface. The Mordell-Weil rank of such a fibration is maximal (=8) because we always have $K^2_{X/\mathbf{P}^1} = 0$ from the theory of elliptic surfaces due to Kodaira [Kod]. When $g \geq 2$, we can also give a series of examples of rational surfaces X with fibrations of curves of genus g whose Mordell-Weil rank is maximal, i.e., $r = 4g + 4$. Hence we see that the upper bound $4g + 4$ is best possible. Let $\pi : \Sigma_e \longrightarrow \mathbf{P}^1$ be the Hirzebruch surface of degree e with $0 \leq e \leq g$. Then we can find a very ample complete linear system whose general members are smooth hyperelliptic curves of genus g. (For detail, see §3). We take a linear Lefschetz pencil of the linear system and obtain a fibration $f : X \longrightarrow \mathbf{P}^1$ by blowing up the base points of the pencil. We can show that such a fibration has the maximal Mordell-Weil rank $4g + 4$ (cf. Proposition 3.7).

Conversely, we can show the following theorem.

Theorem B. *(Cf. Theorem 4.1.) Let* X *be a rational surface with a fibration* $f : X \longrightarrow \mathbf{P}^1$ *of genus* $g \geq 2$. *Assume that the Mordell-Weil rank is maximal, i.e.,* $r = 4g + 4$. *Then* X *is obtained as above, that is,* $f : X \longrightarrow \mathbf{P}^1$ *is obtained as a blowing up of a linear pencil of hyperelliptic curves on the Hirzebruch surface* Σ_e *with* $0 \leq e \leq g$.

(0.3) In [Sh1], [Sh2], Shioda introduced the theory of Mordell-Weil lattices for the fibrations of elliptic curves and also curves of genus $g \geq 2$. In the case of rational elliptic surfaces, Mordell-Weil lattices with maximal rank (= 8) are isometric to the unique even unimodular positive definite lattice of rank 8, E_8. The lattice E_8 plays a very important role as a frame lattice in his theory of Mordell-Weil lattices of the rational elliptic fibration. Even in the higher genus case, we can determine the structures of Mordell-Weil lattices with maximal rank (=4g+4) as a corollary to Theorem B. **Theorem C.** *(Cf. Proposition 3.10.) Let* X *be a rational surface with a fibration* $f : X \longrightarrow \mathbf{P}^1$ *of genus* $g \geq 2$. *Assume that the Mordell-Weil rank is maximal, i.e.,* $r = 4g + 4$. *Then the Mordell-Weil lattice is unique up to isometries. In fact it is a torsion free positive-definite unimodular lattice* L^-_g *whose intersection diagram (i.e., Dynkin diagram) is given in figure 1 in Proposition 3.10.*

We note that L^-_1 is nothing but E_8, hence L^-_g ($g \geq 2$) is a natural generalization

of E_8. Here are ideas which we use in the Proof.s of the above theorems. Theorem A is a consequence of Xiao's inequality [Xiao] and Konno's result [Kon] which gives the affirmative answer to Conjecture 1 in [Xiao]. To prove Theorem B, we use Theorem A and a refinement of Tan's lemma in [Tan] which shows that a rational surface with hyperelliptic fibrations of genus g with maximal Mordell-Weil rank is a double covering of $\mathbf{P}^1 \times \mathbf{P}^1$ whose branch locus is a smooth divisor with bidegree $(2g+2, 2)$. After some birational transformation, we see that such fibrations are obtained by the blowing up of base points of hyperelliptic pencils on Σ_e. Theorem C follows from Theorem B and an explicit calculation of intersection pairings of divisors on surfaces. We are grateful to Professor Konno and Professor Tan for sending us their interesting preprints. We also would like to thank Professor Shioda for helpful and kind comments. Finally, we would like to thank to the referee for useful comments and pointing out mistakes in Theorem 4.1 and Proposition 4.2 in the first version of this paper.

1 Mordell-Weil lattices

We review basic notations and results on Mordell-Weil lattice according to Shioda [Sh1], [Sh2]. Let k be an algebraically closed field and $K = k(C)$ the rational function field of a smooth projective curve C defined over k.

Let Γ/K be a smooth curve of genus $g > 0$ defined over K with a K-rational point $O \in \Gamma(K)$, and let J_Γ/K denote the Jacobian variety of Γ/K. We define the Mordell-Weil group of Γ/K to be the group of K-rational points $J_\Gamma(K)$. Then the Mordell-Weil group is a finitely generated abelian group if the following condition (*) is satisfied (cf. [L]):

(*) The K/k-trace of J_Γ is trivial.

Shioda's main idea in [Sh1] and [Sh2] is to view this Mordell-Weil group $J_\Gamma(K)$ (modulo torsion) as a Euclidean lattice with respect to a natural pairing defined in terms of intersection theory on an associated surface.

Let

$$f : X \longrightarrow C$$

be the relatively minimal fibration of curves associated with given Γ/K. By this, we mean that X is a smooth projective surface, f is a projective morphism with generic fiber Γ/K and there are no exceptional curves of the first kind in any fiber. A K-rational point $P \in \Gamma(K)$ defines a rational section of f, hence defines a regular algebraic section of f. Therefore there is a natural correspondence between the set of K-rational points $\Gamma(K)$ and the set of sections of f, and for $P \in \Gamma(K)$ we write (P) the section regarded as a curve in X.

Let $\mathrm{NS}(X)$ be the Néron-Severi group of X. Then $\mathrm{NS}(X)/torsion$ admits the intersection pairing and Hodge index theorem implies that its signature is $(1, \rho - 1)$

where $\rho = \mathrm{rank}\mathrm{NS}(X)$ is the Picard number of X.

Let T denote the subgroup of $\mathrm{NS}(X)$ generated by (O) and all irreducible components of fibers of f. The sublattice T is called *the trivial lattice*. Then we have the following fundamental result due to Shioda.

Theorem 1.1. *(Cf. [Sh1], [Sh2]). Under the assumption* (*)*, there is a natural isomorphism of groups*

$$J_\Gamma(K) \simeq \mathrm{NS}(X)/T. \tag{1.2}$$

In the following, we also assume that: (**) $\mathrm{NS}(X)$ is torsion-free. This condition is satisfied when X is a rational surface. Let U denote a rank 2 unimodular lattice spanned by (O) and F the class of fiber, and let $\Sigma = \{v \in C(k) |\ f^{-1}(v) \text{ is reducible}\}$. Moreover for each $v \in \Sigma$, we define T_v to be a negative-definite sublattice spanned by the irreducible components of $f^{-1}(v)$ which do not intersect the zero section (O). Then we have the decomposition of the trivial lattice T as follows.

$$T = U \oplus \oplus_{v\in\Sigma} T_v. \tag{1.3}$$

Let us set $r = \mathrm{rank}J_\Gamma(K)$, which we call the Mordell-Weil rank of Γ/K. Then, from (1.2) and (1.3), we have the following formula:

$$r = \rho - 2 - \sum_{v\in\Sigma}(m_v - 1) \tag{1.4}$$

where m_v denotes the number of irreducible components of $f^{-1}(v)$. In particular, if all fibers of f are irreducible, then we have

$$r = \rho - 2. \tag{1.5}$$

Let $L = T^\perp \subset \mathrm{NS}(X)$ be the orthogonal complement of T in $\mathrm{NS}(X)$. Shioda [Sh1], [Sh2] called L *the essential sublattice*, and it is easy to see that L is a negative definite lattice of rank r. We define the dual lattice L^* of L by

$$L^* = \{x \in L \otimes \mathbf{Q} \mid (x, y) \in \mathbf{Z} \ \text{ for all } \ y \in L\},$$

where (x, y) denotes the intersection pairing on $\mathrm{NS}(X)$.

The following lemma and theorem are due to Shioda, and we refer it to [Sh2].

Lemma 1.6. *Under the conditions (*) and (**), there is a unique homomorphism*

$$\phi : J_\Gamma(K) \longrightarrow \mathrm{NS}(X) \otimes \mathbf{Q} \tag{1.7}$$

which splits the isomorphism (1.2), i.e. for any $P \in J_\Gamma(K)$ we have

$$\phi(P) \equiv D_P \ \mathrm{mod} \ \ T \otimes \mathbf{Q}, \quad \phi(P) \perp T$$

where D_P is a horizontal divisor on S corresponding to $P \in J_\Gamma(K) = \mathrm{Pic}^0(\Gamma)(K)$ under (1.2); for instance, we have $D_P = (P)$ for $P \in \Gamma(K) \subset J_\Gamma(K)$. The kernel of this homomorphism is the torsion part $J_\Gamma(K)_{tor}$ of $J_\Gamma(K)$ and we have

$$\mathrm{Im}(\phi) \subset L^*.$$

Theorem 1.8. *Define the symmetric bilinear form on the Mordell-Weil group $J_\Gamma(K)$ by*

$$< P, Q >= -(\phi(P), \phi(Q)) \in \mathbf{Q} \quad (P, Q \in J_\Gamma(K)).$$

Then it induces the structure of a positive-definite lattice on $J_\Gamma(K)/J_\Gamma(K)_{tor}$.

The lattice $(J_\Gamma(K)/J_\Gamma(K)_{tor}, <, >)$ is called *the Mordell-Weil lattice* of Γ/K, or of the fibration $f : X \longrightarrow C$. *The narrow Mordell-Weil lattice* $J_\Gamma(K)^0$ is a sublattice of the Mordell-Weil lattice $J_\Gamma(K)$ such that $J_\Gamma(K)^0 \simeq L/T \subset \mathrm{NS}(X)/T$. We can also define it as a group of sections of the identity component of the Néron model over C of J_Γ/K. It is easy to see that if all fibers of f are irreducible, the narrow Mordell-Weil lattice coincides with the whole Mordell-Weil lattice, i.e. $J_\Gamma(K)^0 \simeq J_\Gamma(K)$.

Theorem 1.9. *The narrow Mordell-Weil lattice $J_\Gamma(K)^0$ is isometric to the opposite lattice L^- of L. Here the opposite lattice of L is a lattice obtained by putting the minus sign on the pairing on L.* **Theorem 1.10.** *Assume that the Néron-Severi lattice* $\mathrm{NS}(X)$ *of X is unimodular and torsion-free, (e.g. X is a rational surface). Then we have the following commutative diagram whose morphisms are natural isometries.*

$$\begin{array}{ccc} J_\Gamma(K)/J_\Gamma(K)_{tor} & \simeq & (L^-)^* \\ \cup & & \cup \\ J_\Gamma(K)^0 & \simeq & L^-. \end{array} \tag{1.11}$$

Theorem 1.12. *Under the assumptions in Theorem 1.10, assume moreover that all fibers of f are irreducible. Then the Mordell-Weil lattice $J_\Gamma(K)$ is a torsion free lattice isometric to the unimodular lattice $J_\Gamma(K)^0 \simeq L^-$ where L is the orthogonal complement of the trivial lattice T generated by the zero section (O) and the class of a general fiber F.*

2 Bounds of Mordell-Weil rank

In this section, we will give an upper bound of Mordell-Weil rank for fibration of curves of genus g on rational surfaces. ¿From §2 to the last, we assume that the base

field k is an algebraically closed field of characteristic zero. The important results we need in this section are Xiao's inequality and Konno's result. Let $f : X \longrightarrow C$ be a relatively minimal fibration of genus $g \geq 1$ over a non-singular projective curve C. Denote $\Delta(f) = \deg f_* \omega_{X/C}$ where $\omega_{X/C}$ is the relative canonical sheaf of f. Assume that f is not locally trivial. Then $\Delta(f) > 0$. Note that the converse is also true. (See for example [BPV, III, Theorem 18.2].)

Let $K_{X/C} = K_X \otimes f^*(K_C^{-1})$ be the relative canonical bundle. We define *the slope* $\lambda(f)$ as the following ratio:

$$\lambda(f) = K_{X/C}^2 / \Delta(f). \tag{2.1}$$

By an easy calculation and the relative Riemann-Roch theorem, we have

$$K_{X/C}^2 = K_X^2 - 8(g(C) - 1)(g - 1), \tag{2.2}$$

$$\Delta(f) = \chi(\mathcal{O}_X) - (g(C) - 1)(g - 1). \tag{2.3}$$

Now the following theorem follows from Xiao's inequality [Xiao] and Konno's result [Kon].

Theorem 2.4. *Let $f : X \longrightarrow C$ be as above and assume that f is not locally trivial and $g \geq 1$. Then we have*

$$\lambda(f) \geq 4(g - 1)/g. \tag{2.5}$$

Moreover suppose that $\lambda(f) = 4(g - 1)/g$. Then $f : X \longrightarrow C$ is a hyperelliptic fibration, i.e., the general fibers of f are hyperelliptic curves.

Proof. When $g \geq 2$, the first assertion is nothing but Theorem 2 in [Xiao]. The second assertion follows from Proposition 2.6 in [Kon], which gives the affirmative answer to Conjecture 1 in [Xiao]. In the case of $g = 1$, since f is relatively minimal,we always have $K_{X/C}^2 = 0$, which implies (2.5). ¿From now on we assume that X is a rational

surface. For such a relatively minimal fibration $f : X \longrightarrow C$ of genus g, C must be the projective line $\mathbf{P}^1$ because $q(X) = 0$. Moreover it is easy to see the following lemma.

Lemma 2.6. *Let $f : X \longrightarrow \mathbf{P}^1$ be a relatively minimal fibration of genus $g \geq 1$ such that X is a rational surface. Then the condition (*) is satisfied, that is, K/k-trace of J_Γ is trivial. Proof.* If the K/k-trace is not trivial, then the Mordell-Weil group is not finitely generated. On the other hand, since the base is a curve $\mathbf{P}^1$, we have the isomorphism

$$\mathrm{Pic}_{X/\mathbf{P}^1}(\mathbf{P}^1) \simeq \mathrm{Pic}(X)/\mathrm{Pic}(\mathbf{P}^1),$$

where $\mathrm{Pic}_{X/\mathbf{P}^1}$ is the relative Picard functor for f. By using theory of the Néron model of J_Γ and its relation to the relative Picard functor, we see that the Mordell-Weil group $J_\Gamma(K)$ is isomorphic to a subquotient of $\mathrm{Pic}_{X/\mathbf{P}^1}(\mathbf{P}^1)$. (We refer these to [9.5, BLR].) Since X is a rational surface, $\mathrm{Pic}(X)$ is isomorphic to $\mathrm{NS}(X)$ which is a finitely generated abelian group. Therefore we see that $J_\Gamma(K)$ is finitely generated.

For a fibration $f : X \longrightarrow \mathbf{P}^1$ of genus $g \geq 1$ such that X is a rational surface, we can easily see that

$$\Delta(f) = \chi(\mathcal{O}_X) - (0-1)(g-1) = 1 + (g-1) = g. \tag{2.7}$$

Therefore we have $\Delta(f) = g > 0$, which shows that f is not locally trivial.

Theorem 2.8. *Let X be a rational surface with a fibration $f : X \longrightarrow \mathbf{P}^1$ of curves of genus $g \geq 1$ which is relatively minimal. Let $J_\Gamma(K)$ be the Mordell-Weil group of this fibration and $r = \mathrm{rank} J_\Gamma(K)$ its Mordell-Weil rank. Then we have*

$$r \leq 4g + 4. \tag{2.9}$$

Moreover we have $r = 4g + 4$ if and only if $f : X \longrightarrow \mathbf{P}^1$ is a hyperelliptic fibration with $K^2_{X/\mathbf{P}^1} = 4g - 4$ such that all fibers of f are irreducible.

Proof. Since X is a rational surface, the Picard number $\rho(X)$ is equal to $b_2(X) = \dim H^2(X, \mathbf{C})$. Since $b_1(X) = 0$, Noether's formula and (2.2) imply that

$$\begin{aligned} \rho(X) &= 12\chi(\mathcal{O}_X) - (K^2_{X/\mathbf{P}^1} - 8(g-1)) - 2, \\ &= 8g + 2 - K^2_{X/\mathbf{P}^1}. \end{aligned}$$

On the other hand, since f is not locally trivial we can apply the slope inequality (2.5), which implies that $K^2_{X/\mathbf{P}^1} \geq 4g - 4$ because $\Delta(f) = g$. Hence we have $\rho(X) \leq 4g + 6$. This and the formula (1.4) imply that $r \leq \rho(X) - 2 \leq 4g + 4$. Moreover the equality $r = 4g + 4$ holds if and only if $K^2_{X/\mathbf{P}^1} = 4g - 4$ and all fibers of f are irreducible. Therefore we have the rest of the assertions from Theorem 2.4.

3 Examples of fibrations with the maximal Mordell-Weil rank

In this section we shall construct examples of rational surfaces with fibration of genus $g \geq 1$ whose Mordell-Weil ranks are maximal.

Let $\pi : \Sigma_e = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(e)) \longrightarrow \mathbf{P}^1$ be the Hirzebruch surface of degree e with $0 \leq e \leq g$. The Picard group $\mathrm{Pic}(\Sigma_e)$ or $\mathrm{NS}(\Sigma_e)$ is generated by the classes of a

tautological section C_∞ and a fiber F_0 of π. The intersection pairings on $\mathrm{NS}(\Sigma_e)$ are given as follows:

$$C_\infty^2 = e, \quad (C_\infty \cdot F_0) = 1, \quad (F_0)^2 = 0.$$

The minimal section C_0 of Σ_e is equal to $C_\infty - eF_0$ in $\mathrm{NS}(\Sigma_e)$, hence $C_0^2 = -e$. First we have the following easy lemma.

Lemma 3.1. *Set $a = g + 1 - e > 0$. Then the linear system $|2C_\infty + aF_0|$ is very ample. Hence a general member D of $|2C_\infty + aF_0|$ is a non-singular irreducible hyperelliptic curve of genus g.*

Proof. Since $a = g + 1 - e > 0$, the first part follows from [Cor. 2.18, V, [H]], and it implies that the existence of a nonsingular irreducible member D. Since a natural projection $D \longrightarrow \mathbf{P}^1$ is a 2-1 map, D is a hyperelliptic curve. Noting that

$$K_{\Sigma_e} = -2C_\infty + (e-2)F_0, \tag{3.2}$$

we have

$$g(D) = (K_{\Sigma_e} + D) \cdot D/2 + 1 = (g-1)F_0 \cdot D/2 + 1 = g.$$

Moreover from the very ampleness of the linear system $|2C_\infty + aF_0|$, we can find its generic smooth irreducible members D_0 and D_1 which give *a Lefschetz pencil* on Σ_e. By this we mean that the linear pencil $\{D_t\}_{t \in \mathbf{P}^1}$ given by D_0 and D_1 satisfies the following conditions: (3.3)

> Most of members D_t are smooth and every members in the pencil
>
> is irreducible and has at most one node as its singularity.

For the existence of Lefschetz pencil, see [SGA 7 II, Exposé XVII].

Next note that $D_0 \cdot D_1 = (2C_\infty + aF_0)^2 = 4e + 4a = 4g + 4$ and we can assume that D_0 and D_1 intersects each other transversely. Therefore we have 4g+4 distinct points $p_1, \cdots, p_{4g+4}$ which are the base points of the pencil. (We may also assume that they do not lie on the minimal section C_0 and any two of them are not on the same fiber of π.)

Under these assumption, let $\phi : X \longrightarrow \Sigma_e$ be the blowing up of the points $p_1, \cdots, p_{4g+4}$, then we obtain the fibration $f : X \longrightarrow \mathbf{P}^1$ of curves of genus g. Summarizing the above results, we have the following proposition. **Proposition 3.4.** *The fibration $f : X \longrightarrow \mathbf{P}^1$ obtained as above is a hyperelliptic fibration of genus g which is not locally trivial. Moreover all fibers are irreducible and every singular fiber has at most one node as its singularity.*

Let F denote the class of a fiber of f, and $E_i = \phi^*(p_i)$ the exceptional curve dominating the point p_i. For simplicity, we also denote the total transforms of C_∞, C_0, F_0 by the blowing up ϕ by the same letters.

Then the Néron-Severi group $\mathrm{NS}(X)$ is isomorphic to the free module

$$\mathrm{NS}(X) \simeq \mathbf{Z} \cdot C_\infty \oplus \mathbf{Z} \cdot F_0 \oplus (\oplus_{i=1}^{4g+4} \mathbf{Z} \cdot E_i). \tag{3.5}$$

Moreover in the Néron-Severi group $\mathrm{NS}(X)$, we have the relation:

$$F = 2C_\infty + aF_0 - \sum_{i=1}^{4g+4} E_i. \tag{3.6}$$

Let $K = k(\mathbf{P}^1)$ be the rational function field of $\mathbf{P}^1_k$ and let Γ/K denote the generic fiber of $f : X \longrightarrow \mathbf{P}^1$. Since all fibers of f are irreducible (Theorem 3.4), the narrow Mordell-Weil group $J_\Gamma(K)^0$ coincides with the whole Mordell-Weil group $J_\Gamma(K)$ (Theorem 1.12).

Proposition 3.7. *For a fibration $f : X \longrightarrow \mathbf{P}^1$ of genus g in Proposition 3.4, we have*

$$K^2_{X/\mathbf{P}^1} = 4g - 4.$$

Hence $f : X \longrightarrow \mathbf{P}^1$ is a fibration of curves of genus g whose Mordell-Weil rank is maximal, i.e., equal to $4g + 4$.

Proof. Since from (3.2) we have

$$K_X = \phi^*(K_{\Sigma_e}) + \sum_{i=1}^{4g+4} E_i = -2C_\infty + (e-2)F_0 + \sum_{i=1}^{4g+4} E_i,$$

we obtain

$$K_{X/\mathbf{P}^1} = K_X + 2F = 2C_\infty + (2a + e - 2)F_0 - \sum_{i=1}^{4g+4} E_i.$$

Therefore we have

$$K^2_{X/\mathbf{P}^1} = 4e + 4(2a + e - 2) - (4g + 4) = 4g - 4.$$

Since all fibers of f are irreducible (Proposition 3.4), the rest of the assertions follow from this and Theorem 2.8. Q.E.D. **Definition 3.8.** A fibration $f : X \longrightarrow \mathbf{P}^1$ of genus g constructed from the blowing up of Σ_e with $0 \leq e \leq g$ as above is called a fibration of type (g, e).

Now we shall determine the structure of the Mordell-Weil lattice $(J_\Gamma(K), <, >)$ of a fibration $f : X \longrightarrow \mathbf{P}^1$ of type (g, e). Since $(F, E_i) = 1$, the rational curves $\{E_i\}$

become sections of f, and we take E_1 as the zero section (O). Then by definition and Proposition 3.4, the trivial sublattice $T = T_{g,e} \subset \mathrm{NS}(X)$ is generated by the class of E_1 and the fiber F of f. ¿From Theorem 1.1, 1.12 and Proposition 3.4, we obtain isomorphisms of groups:

$$J_\Gamma(K)^0 \simeq J_\Gamma(K) \simeq \mathrm{NS}(X)/T. \tag{3.9}$$

Moreover from 1.12 and 3.4, the Mordell-Weil lattice $(J_\Gamma(K), <,>)$ is isomorphic to $L^-_{g,e}$ where $L_{g,e}$ is the orthogonal complement of the trivial lattice $T_{g,e}$ as in §1. The following proposition determines the structure of the lattice $L^-_{g,e}$.

Proposition 3.10. *For $g \geq 1$ and $0 \leq e \leq g$, the lattice $L^-_{g,e}$ is a positive-definite unimodular lattice of rank $4g+4$ whose Dynkin diagram is given as follows.*

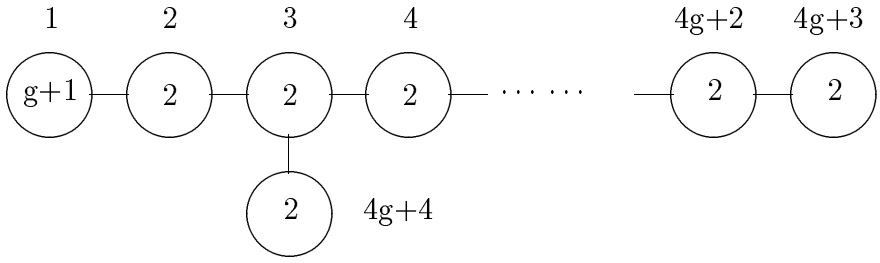

Figure 1.

Here the numbers in the circles denote the self intersections of elements, and a line between two circles shows that the paring of two elements is equal to -1.

Moreover $L^-_{g,e}$ is an even (resp. odd) lattice if g is odd (resp. even). Proof. We take an integer m as

$$\begin{array}{ll} 2m+1 = g+e+1 & \text{if } g+e+1 \text{ is odd,} \\ 2m = g+e+1 & \text{if } g+e+1 \text{ is even.} \end{array}$$

Since $T_{g,e}$ is generated by two elements:

$$F = 2C_\infty + (g+1-e)F_0 - \sum_{i=1}^{4g+4} E_i, \quad E_1,$$

it is easy to see that the following elements form basis of $L^-_{g,e}$ in each case.
Case $g+e+1 = 2m+1$:

$$\begin{array}{ccc} H_1 = C_\infty - mF_0 - E_2, & H_2 = E_2 - E_3, & H_3 = E_3 - E_4, \\ \cdots, & H_{4g+3} = E_{4g+3} - E_{4g+4}, & H_{4g+4} = F_0 - E_2 - E_3. \end{array}$$

Case $g+e+1=2m$**:**

$$\begin{array}{ccc} H_1 = C_\infty - mF_0, & H_2 = F_0 - E_2 - E_3, & H_3 = E_3 - E_4, \\ \cdots, & H_{4g+3} = E_{4g+3} - E_{4g+4}, & H_{4g+4} = E_2 - E_3. \end{array}$$

(The numbers of elements correspond to those in figure 1.)

Then taking the minus sign on the pairing on $L^-_{g,e}$ into account, we can easily check that the intersection matrix is given by the Dynkin diagram in figure 1 and all other statements follow from this.

Definition 3.11. ¿From Proposition 3.10, we see that the structure of the lattice $L_{g,e}$ depends only on g, so we denote it by L_g. Hence L^-_g is the positive-definite unimodular lattice of rank $4g+4$ whose Dynkin diagram is given by figure 1.

Remark 3.12. Professor Shioda pointed out to us that the fibrations $f : X \longrightarrow \mathbf{P}^1$ in this section can be obtained as special cases of his examples in [Theorem 3, Sh3].

4 Uniqueness of the maximal Mordell-Weil lattice

In this section, we prove the following theorem. **Theorem 4.1.** *Let X be a rational surface with a fibration $f : X \longrightarrow \mathbf{P}^1$ of curves of genus $g \geq 2$ which is relatively minimal. Assume that the Mordell-Weil rank r of f is maximal, i.e., $r = 4g+4$. Then $f : X \longrightarrow \mathbf{P}^1$ is a fibration of type (g,e) described in §3, that is, it is obtained as a blowing up of a pencil of hyperelliptic curves on the Hirzebruch surface Σ_e with $0 \leq e \leq g$. In particular, if X is rational and f is relatively minimal, the Mordell-Weil lattice arising from a fibration of curves genus $g \geq 2$ with maximal Mordell-Weil rank is always isometric to L^-_g defined in Proposition 3.11.*

We first recall the following proposition. **Proposition 4.2.** *Let X be as in Theorem 4.1. Then X is a double covering of $\mathbf{P}^1 \times \mathbf{P}^1$ branched along a smooth curve B of type $(2g+2, 2)$. Moreover the fibration $f : X \longrightarrow \mathbf{P}^1$ is induced by the second projection p_2.*

Proof. ¿From Theorem 2.8, we infer that $f : X \longrightarrow \mathbf{P}^1$ is a rational hyperellitic fibration of genus g with the minimal slope $\lambda(f) = 4 - 4/g$ (or equivalently $K^2_{X/\mathbf{P}^1} = 4g-4$ and all of fibers f are irreducible. Then we recall the following fact. If $f : X \longrightarrow \mathbf{P}^1$ is a hyperellitic fibration of genus g with the minimal slope $\lambda(f) = 4-4/g$, there exists a Hirzebruch surface Σ_e and a double covering $\tau : Y \longrightarrow \Sigma_e$, such that Y has only rational double points as its singularities, and X is the minimal resolution

of Y. Moreover the fibration f is induced by the ruling of $\Sigma_e \longrightarrow \mathbf{P}^1$. For these facts, we refer to the Proof. of [Theorem 2.1, Ho] or [Proposition 2.12, P]. We remark that the slope inequality (2.5) is equivalent to the inequality in [Theorem 2.1, Ho] and [Proposition 2.12, P] and the equality holds for the canonical resolution $\tilde{Y}$ of Y if and only if all of singularities of the standardized branch locus B of $\tau : Y \longrightarrow \Sigma_e$ in [Theorem 2.1, Ho] have multiplicities equal to 2 or 3. The later fact implies that Y has only rational double points as its singularities and $\tilde{Y}$ is the minimal resolution of Y. Since the original X can be obtained by contracting (-1)-curves on $\tilde{Y}$ contained in fibers, if $\lambda(f) = 4 - 4/g$ we see that $\tilde{Y} = X$.

Next we recall the argument of the Proof. of [Lemma 3.2, Tan]. Let $\tau : Y \longrightarrow \Sigma_e$ be a double covering whose branch locus B is linearly equivalent to $(2g+2)C_0+2mF_0$ where C_0, F_0 are given as in §3. Since Y has only rational double points as its singularities, we have $(K_{Y/\mathbf{P}^1})^2 = (K_{X/\mathbf{P}^1})^2$ and $\chi(\mathcal{O}_\mathcal{Y}) = \chi(\mathcal{O}_\mathcal{X}) = \}$, we must have

$$(K_{Y/\mathbf{P}^1})^2 = (K_Y)^2 + 8(g-1) = (K_{X/\mathbf{P}^1})^2 = 4g-4, \text{ or } (K_Y)^2 = -4(g-1).$$

Since $K_{\Sigma_e} = -2C_0 - (e+2)F_0$ and

$$K_Y = \tau^*(K_{\Sigma_e} + (1/2)B) = \tau^*((g-1)C_0 + (m-e-2)F_0), \tag{4.2.1}$$

we have

$$(K_Y)^2 = 2(-(g-1)^2e+2(g-1)(m-e-2)) = -2(g-1)((g+1)e-2m+4) = -4(g-1).$$

Since $g > 1$, from this we have $(g+1)e = 2(m-1)$. On the other hand, from (4.2.1), we have $m < e+2$ because Y is a rational surface. These imply that $e = 0$ and $m = 1$, or equivalently Y is a double covering of $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$ whose branch locus B is of type $(2g+2, 2)$. It remains to show that B is smooth which also implies that $X = Y$. If B is not smooth, there is a rational curve arising from the resolution of singularities which lies on a fiber of f. This contradicts to the fact that all fibers of f are irreducible.

Remark 4.3. In the first version of this paper, we stated that Proposition 4.2 is true even if $g = 1$. As the referee pointed out to us, Proposition 4.2 is not true in the case of $g = 1$. We have two more cases for (e, m), that is, $(1, 2), (2, 3)$. (These cases really occur.) The case $(e, m) = (2, 3)$ whose branch locus is $C_0 + B'$ where $B' \in |3C_\infty|$ corresponds to the Weierstrass fibration of a rational elliptic surface. Note that all rational elliptic fibrations with fixed sections are the minimal resolutions of the Weierstrass fibrations. Moreover if all fibers of an elliptic surface are irreducible, it is isomorphic to a Weierstrass fibration.

Now we prove Theorem 4.1. Let $f : X \longrightarrow \mathbf{P}^1$ be a fibration in Theorem 4.1. Then from Proposition 4.2, we obtain a double covering $\tau : X \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$ whose

branch locus $B \subset \mathbf{P}^1 \times \mathbf{P}^1$ is a smooth curves of type $(2g+2, 2)$. Restricting the first projection p_1 of $\mathbf{P}^1 \times \mathbf{P}^1$ to B, we have the double covering $p_{1|B} : B \longrightarrow \mathbf{P}^1$. It is easy to see that the genus of B is $2g+1$, hence there are $4g+4$ distinct branched points of $p_{1|B}$. Let $\varphi : X \longrightarrow \mathbf{P}^1$ be a fibration induced from p_1. Then this is a conic bundle with $4g+4$ reducible conics over the branch points of $p_{1|B}$. Let $\{E_i^+, E_i^-\}_{i=1}^{4g+4}$ be irreducible components of these reducible conics such that $\varphi(E_i^+) = \varphi(E_i^-)$. It is easy to see that each curve $E_i^{\pm}$ is a (-1)-rational curve, hence for each $1 \leq i \leq 4g+4$, we can contract one of $E_i^{\pm}$s and obtain a smooth ruled surface $\pi : S \longrightarrow \mathbf{P}^1$.

Since all fibers of π are $\mathbf{P}^1$, $\pi : S \longrightarrow \mathbf{P}^1$ is isomorphic to a Hirzebruch surface $\pi : \Sigma_e \longrightarrow \mathbf{P}^1$ of degree e for some e. Hence we have a birational morphism $\phi : X \longrightarrow \Sigma_e$ by contracting one of $E_i^{\pm}$s for all i. Let $F \subset X$ denote a smooth general fiber of f, and set $F' = \phi(F) \subset \Sigma_e$. Since $E_i^{\pm}$s are sections of f, we may assume that F' is smooth and birational to F. Using the same notation as in §3, we set

$$F' = \alpha C_\infty + \beta F_0.$$

It is easy to see that $(F', F_0) = 2$, hence we have $\alpha = 2$. By the same calculation as in Lemma 3.1, we have

$$g = g(F) = g(F') = ((e+\beta-2)F_0, 2C_\infty + \beta F_0)/2 + 1 = e + \beta - 1.$$

or

$$\beta = g + 1 - e.$$

(Cf. Lemma 3.1.) Now setting $a = \beta = g + 1 - e$, we proved that F' belongs to the linear system $|2C_\infty + aF_0|$ and $\phi : X \longrightarrow \Sigma_e$ is the blowing up of base points of a linear pencil in this linear system. It remains to show that $a = g + 1 - e$ is positive. Otherwise, $(F', C_0) = a$ is non-positive where C_0 is the minimal section. If $a < 0$, then C_0 is in the base locus of the pencil, which contradicts to the fact the base locus of the pencil is zero dimensional. If $a = 0$, let C_0' be the proper transform of C_0 by ϕ. It is easy to see that $(F, C_0') = 0$ and $C_0^2 = (C_0')^2 = -e = -g - 1$. Hence C_0' is an irreducible component of a fiber of f, and since all fibers of f are irreducible we have $(C_0')^2 = 0 \neq -g - 1$. This is a contradiction.

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