Q-STRUCTURE OF CONFORMAL FIELD THEORY WITH GAUGE SYMMETRIES

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To the memory of Osamu Hyodo

Introduction

It is well-known that abelian conformal field theory has rich arithmetic structure (see, for example [DKK], [KSU1], [KSU2], [KUS3]). It is natural to ask whether this is also the case for no-abelian conformal field theory. For the first step to study arithmetic properties of non-abelian conformal field theory, in the present notes we shall study the field of definition of conformal field theory. Namely we shall show that conformal field theory with gauge symmetries developed in [TUY] can be defined over the rational number field $\mathbf{Q}$. This means that the sheaf of vacua (or conformal blocks) over the moduli space of pointed stable curves with formal neighbourhoods is defined over $\mathbf{Q}$ and the projectively flat connection on it is also defined over $\mathbf{Q}$. More precisely, if a family $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \dots, s_N; \eta_1, \dots, \eta_N)$ of N-pointed stable curves with formal neighbourhoods is defined over a field K of characteristic zero, then the sheaf of vacua $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F})$ is defined over the field K and the projective flat connection on it is also defined over K. This agrees with the fact that the space of vacua is isomorphic to the space of generalized theta functions $H^0(\mathfrak{M}, \mathcal{L}^{\ell})$ which is defined over $\mathbf{Q}$. (See, for example [BL].)

Once conformal field theory is defined over $\mathbf{Q}$, it is natural to consider the reduction modulo p for each prime p. For almost all prime p, we have conformal field theory over a field of characteristic p. To study such a theory a natural approach is to localize the theory and study conformal field theory over a discrete valuation ring. This will be discussed in the forthcoming paper. To develop such a theory, the results in the present notes play the crucial role.

In the present notes we follow operator formalism of [TUY] to develop conformal field theory over $\mathbf{Q}$. Unfortunately, there are many misprints and even some mistakes in [TUY]. The serious mistake is in the proof of Proposition 5.1.7. There,

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we implicitly used local-freeness of the sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ before proving the local-freeness. Logical order of the arguments in [TUY] should be as follows. First we can prove the local-freeness of the sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(\infty)})$ and the sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}^{(1)})$ by using Proposition 5.1.3. Therefore, Theorem 6.2.1(local-freeness of the sheaf of vacua) can be proved without using Proposition 5.1.7. Instead we can use Theorem 6.2.1 to prove proposition 5.1.7 and its consequence. In this way we can prove all the results in [TUY] by changing order of the arguments in [TUY]. These arguments is outlined in [U1] and the details can be found in [U2].

Let us explain briefly the contents of the present notes. in §1 we shall recall basic properties of $\mathbf{Q}$ -structure of an affine lie algebra associated with a simple Lie algebra and its highest weight integrable modules. In §2 we shall define the vacua (or conformal blocks) attached to an N-pointed stable curves with formal neighbourhoods defined over a field K of characteristic zero and its sheaf version. They are defined over the field K. Also we shall prove their basic properties based on the results in [TUK]. In §3 we shall show how to define the projectively flat connection with regular singularities along the boundaries on the sheaf of vacua attached to a family of N-pointed stable curves with formal neighbourhoods defined over the field K.

§1 Affine Lie algebra

Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers $\mathbf{C}$ and $\mathfrak{h}$ its Cartan subalgebra. By Δ we denote the root system of $(\mathfrak{g}, \mathfrak{h})$. We have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_{\alpha}. \tag{1.1}$$

Fix a lexicographic ordering of $\mathfrak{h}^*_{\mathbf{R}}$ once for all. This gives the decomposition $\Delta = \Delta_+ \sqcup \Delta_-$ of the root system into the positive roots and the negative roots. Let θ be the maximal root. We normalize the Cartan-Killing form

$$(\quad , \quad) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbf{C}$$

in such a way that we have

$$(\theta, \theta) = 2.$$

Existence of Chevalley basis (see for example [H, 25.2]) implies that there exists a simple Lie algebra $\mathfrak{g}(\mathbf{Q})$ over the rational number field $\mathbf{Q}$ such that

$$\mathfrak{g} = \mathfrak{g}(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$$

and that Cartan subalgebra $\mathfrak{h}$ and the root space decomposition (1.1) are also defined over $\mathbf{Q}$. That is, there exist a $\mathbf{Q}$-subalgebra $\mathfrak{h}(\mathbf{Q})$ and $\mathbf{Q}$-subspace $\mathfrak{g}_{\alpha}(\mathbf{Q})$ of $\mathfrak{g}(\mathbf{Q})$ with

$$\mathfrak{h}(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} = \mathfrak{h}, \quad \mathfrak{g}_{\alpha}(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} = \mathfrak{g}_{\alpha}.$$

Hence, the normalized Cartan-Killing form $(\ ,\ )$ is $\mathbf{Q}$-valued over $\mathfrak{g}(\mathbf{Q})$.

The affine algebra $\widehat{\mathfrak{g}}(\mathbf{Q})$ over $\mathbf{Q}$ is defined as

$$\widehat{\mathfrak{g}}(\mathbf{Q}) = \mathfrak{g}(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}((\xi)) \oplus \mathbf{Q} \cdot c$$

where $\mathbf{Q}((\xi))$ is the field of formal Laurent series with coefficients in $\mathbf{Q}$ and

1) c belongs to the center of $\widehat{\mathfrak{g}}(\mathbf{Q})$;

2) the Lie algebra structure is given by

$$[X \otimes f(\xi), Y \otimes g(\xi)] = [X, Y] \otimes f(\xi)g(\xi) + (X, Y) \operatorname*{Res}_{\xi=0}(g(\xi)df(\xi)) \cdot c.$$

For any filed K of characteristic zero put

$$\mathfrak{g}(K) = \mathfrak{g}(\mathbf{Q}) \otimes_{\mathbf{Q}} K, \quad \widehat{\mathfrak{g}}(K) = \widehat{\mathfrak{g}}(\mathbf{Q}) \otimes_{\mathbf{Q}} K.$$

Put

$$\widehat{\mathfrak{g}}_+(\mathbf{Q}) = \mathfrak{g}(\mathbf{Q}) \otimes \mathbf{Q}[[\xi]]\xi, \quad \widehat{\mathfrak{g}}_-(\mathbf{Q}) = \mathfrak{g}(\mathbf{Q}) \otimes \mathbf{Q}[\xi^{-1}]\xi^{-1}.$$

We also identify $\mathfrak{g}(\mathbf{Q})$ with $\mathfrak{g} \otimes 1$ and regard it as a $\mathbf{Q}$-Lie subalgebra of $\widehat{\mathfrak{g}}(\mathbf{Q})$.

Fix a positive integer ℓ (called the *level*) and put

$$P_\ell = \{\, \lambda \in P_+ \mid 0 \leq (\theta, \lambda) \leq \ell \,\}.$$

Proposition 1.1. *For each $\lambda \in P_\ell$ there exists an irreducible left $\widehat{\mathfrak{g}}(\mathbf{Q})$-module $\mathcal{H}_\lambda(\mathbf{Q})$ (called the* integrable highest weight $\widehat{\mathfrak{g}}(\mathbf{Q})$-module*) satisfying the following properties.*

(1) *$V_\lambda(\mathbf{Q}) = \{\, |v\rangle \in \mathcal{H}_\lambda(\mathbf{Q}) \mid \widehat{\mathfrak{g}}_+(\mathbf{Q})|v\rangle = 0 \,\}$ is the irreducible left $\mathfrak{g}(\mathbf{Q})$-module with highest weight λ.*

(2) *The central element c acts on $\mathcal{H}_\lambda(\mathbf{Q})$ as $\ell \cdot \mathrm{id}$.*

(3) *$\mathcal{H}_\lambda(\mathbf{Q})$ is generated by $V_\lambda(\mathbf{Q})$ over $\widehat{\mathfrak{g}}(\mathbf{Q})$ with only one relation*

$$(X_\theta \otimes \xi^{-1})^{\ell-(\theta,\lambda)+1}|\lambda\rangle = 0$$

where $X_\theta \in \mathfrak{g}_\theta(\mathbf{Q})$ is a non-zero element and $|\lambda\rangle \in V_\lambda(\mathbf{Q})$ is the highest weight vector.

Put

$$\mathcal{H}_\lambda^\dagger(\mathbf{Q}) := \mathrm{Hom}_{\mathbf{Q}}(\mathcal{H}_\lambda(\mathbf{Q}), \mathbf{Q}).$$

Then, $\mathcal{H}_\lambda^\dagger(\mathbf{Q})$ is an integrable highest weight right $\widehat{\mathfrak{g}}(\mathbf{Q})$-module. For any field K of characteristic zero put

$$\mathcal{H}_\lambda(K) := \mathcal{H}_\lambda(\mathbf{Q}) \otimes_{\mathbf{Q}} K, \quad \mathcal{H}_\lambda^\dagger(K) := \mathcal{H}_\lambda^\dagger(\mathbf{Q}) \otimes_{\mathbf{Q}} K.$$

Then, $\mathcal{H}_\lambda(\mathbf{C})$ is isomorphic to the usual highest integrable $\widehat{\mathfrak{g}}(\mathbf{C})$-module of level ℓ ([Kac, Chap. 9]). In the following we identify them and we may regard $\mathcal{H}_\lambda(\mathbf{Q})$ as a subset of $\mathcal{H}_\lambda(\mathbf{C})$. The above proposition is an easy consequence of this fact. We also have the canonical bilinear pairing

$$\begin{aligned} \mathcal{H}^\dagger_\lambda(K) \times \mathcal{H}_\lambda(K) &\to K \\ (\langle\Phi|, |\Psi\rangle) &\mapsto \langle\Phi|\Psi\rangle. \end{aligned} \tag{1.2}$$

In the following, for each $\lambda \in P_\ell$, we fix a highest weight vector $|\lambda\rangle \in \mathcal{H}_{\vec{\lambda}}(\mathbf{Q})$ and $\langle\lambda| \in \mathcal{H}^\dagger_{\vec{\lambda}}(\mathbf{Q})$ with

$$\langle\lambda|\lambda\rangle = 1.$$

Also note that for each subfield $K \subset \mathbf{C}$ we have a natural inclusion

$$\begin{aligned} \mathcal{H}_{\vec{\lambda}}(K) &\subset \mathcal{H}_{\vec{\lambda}}(\mathbf{C}) \\ \mathcal{H}^\dagger_{\vec{\lambda}}(K) &\subset \mathcal{H}^\dagger_{\vec{\lambda}}(\mathbf{C}) \end{aligned}$$

Almost all arguments in [TUY, §1] can be valid over $\mathbf{Q}$. But we need to modify the definition of the energy-momentum tensor in the following way. Let $\{X_1, \dots, X_{\dim \mathfrak{g}}\}$ be a $\mathbf{Q}$-basis of $\mathfrak{g}(\mathbf{Q})$ and $\{X^1, \dots, X^{\dim \mathfrak{g}}\}$ be the dual basis with respect to the normalized Cartan-Killing form. Then, the energy-momentum tensor $T(z)$ is defined as

$$T(z) = \frac{1}{2(g^* + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} {}^{\circ}_{\circ} X_a(z) X^a(z) {}^{\circ}_{\circ}$$

where g^* is the dual Coxeter number of $\mathfrak{g}$ and ${}^{\circ}_{\circ}\ \ {}^{\circ}_{\circ}$ is the normal ordering defined by

$${}^{\circ}_{\circ} X(n)Y(m) {}^{\circ}_{\circ} = \begin{cases} X(n)Y(m), & n < m, \\ \frac{1}{2}(X(n)Y(m) + Y(m)X(n)) & n = m, \\ Y(m)X(n) & n > m. \end{cases}$$

Put

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}.$$

Then, $\{L_n\}$ forms a Virasoro algebra defined over $\mathbf{Q}$ with central charge $c_v = \ell \dim \mathfrak{g}/(g^* + \ell)$. The rest of the arguments in 1.2 and 1.3 in [TUY, §1] are valid over $\mathbf{Q}$. In particular we can define a subspace $\mathcal{H}_\lambda(d)(\mathbf{Q})$ by

$$\mathcal{H}_\lambda(d)(\mathbf{Q}) = \{\ |\psi \in \mathcal{H}_\lambda(\mathbf{Q})\rangle \mid L_0|\psi\rangle = (d + \Delta_\lambda|\psi\rangle\ \}$$

where

$$\Delta_\lambda = \frac{(\lambda,\lambda)+2(\lambda,\rho)}{2(g^*+\ell)}, \quad \rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha.$$

The filtration of $\mathcal{H}_\lambda(\mathbf{Q})$ is defined by

$$F_p\mathcal{H}_\lambda(\mathbf{Q}) = \sum_{d=0}^{p}\mathcal{H}_\lambda(d)(\mathbf{Q}). \tag{1.3}$$

By taking the tensor product of a field K of characteristic zero over $\mathbf{Q}$ to (1.3) we can introduce the filtration of $\mathcal{H}_\lambda(K)$. The filtration on $\mathcal{H}^\dagger_\lambda(K)$ is defined by

$$F^p\mathcal{H}^\dagger_\lambda = \prod_{d\geq p}\mathcal{H}^\dagger_\lambda(d)(K)$$

where we define

$$\mathcal{H}_\lambda(d)(K) = \mathcal{H}_\lambda(d)(\mathbf{Q})\otimes_{\mathbf{Q}} K, \quad \mathcal{H}^\dagger_\lambda(d)(K) = \mathrm{Hom}_K(\mathcal{H}_\lambda(K), K).$$

On the other hand the correspondence between Aut $\mathbf{C}[[\xi]]$ and the corresponding Lie algebra $\mathbf{C}[[\xi]]\dfrac{d}{d\xi}$ in 1.4 of [TUY, §1] has no good theory over $\mathbf{Q}$. But as we shall explain in §3 we can avoid the difficulty arising from this fact.

§2 Vacua and Sheaf of Vacua

Let K be a field of characteristic zero. Let $\mathfrak{X} = (C;\, Q_1, Q_2, \ldots, Q_N)$ be an N-pointed stable curve defined as in [TUY, §2]. If the curve C is defined over K and Q_j is a K-rational point of C for all j's, the N-pointed stable curve $\mathfrak{X}$ is called to be defined over K. More precisely, for $\mathfrak{X} = (C;\, Q_1, Q_2, \ldots, Q_N)$ consisting of K-scheme C and K-rational points Q_j of C is called an N-pointed K-stable curve, if $\mathfrak{X}\otimes_K \mathbf{C}$ is an N-pointed stable curve. $(C; Q_1, Q_2, \ldots, Q_N; \eta_1^{(n)}, \eta_2^{(n)}, \ldots, \eta_N^{(n)})$ is called an *N-pointed K-stable curve of genus g with n-th infinitesimal neighbourhoods defined over K*, if $(C; Q_1, Q_2, \ldots, Q_N)$ is an N-pointed K-stable curve and $\eta_j^{(n)}$ is a K-algebra isomorphism

$$\eta_j^{(n)} : \mathcal{O}_{C,Q_j}/\mathfrak{m}_{Q_j}^{n+1} \simeq K[[\,\xi\,]]/(\xi^{n+1}) \tag{2.1}$$

where $\mathfrak{m}_{Q_j}$ is the maximal ideal of $\mathcal{O}_{C,Q_j}$ consisting of germs of regular functions vanishing at Q.

Similarly, we define an N-pointed stable curves of genus g with formal neighbourhoods defined over K by changing the condition (2.1) by

$$\eta_j : \widehat{\mathcal{O}}_{C,Q_j} = \varprojlim_n \mathcal{O}_{C,Q_j}/\mathfrak{m}_{Q_j}^n \simeq K[[\xi]].$$

Let $\mathfrak{X}_K = (C; Q_1, Q_2, \dots, Q_N; \eta_1, \eta_2, \dots, \eta_N)$ be an N-pointed K-stable curve with formal neighbourhoods defined over a field K of characteristic zero. To $\mathfrak{X}_K$ we associate the space of vacua (conformal blocks) $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}_K)$ and the space of covacua $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}_K)$ where $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$, $\lambda_j \in P_\ell$. For that purpose we first define a generalized affine Lie algebra $\widehat{\mathfrak{g}}_N(K)$. Let $\mathfrak{g}$ be a simple Lie algebra with $\mathbf{Q}$-structure given by Chevalley basis. Let $K((\xi))$ be the field of Laurent series of variable ξ with coefficients in the field K. A Lie algebra $\widehat{\mathfrak{g}}_N(K)$ is defined as

$$\widehat{\mathfrak{g}}_N(K) = \bigoplus_{j=1}^N \mathfrak{g}(K) \otimes_K K((\xi_j)) \oplus Kc$$

with the following commutation relations.

$$[(X_j \otimes f_j), (Y_j \otimes g_j)] = ([X_j, Y_j] \otimes f_j g_j) + c \sum_{j=1}^N (X_j, Y_j) \operatorname*{Res}_{\xi_j = 0}(g_j df_j)$$

where (a_j) means $(a_1, a_2, \dots, a_N)$ and c belongs to the center of $\widehat{\mathfrak{g}}_N(K)$.

We also put

$$\widehat{\mathfrak{g}}(\mathfrak{X}_K) = \mathfrak{g}_K \otimes_K H^0(C, \mathcal{O}_{\mathcal{C}}(* \sum_{j=1}^N Q_j)).$$

We have a natural embedding

$$t : H^0(C, \mathcal{O}_{\mathcal{C}}(* \sum_{j=1}^N Q_j)) \hookrightarrow \bigoplus_{j=1}^N K((\xi_j)). \tag{2.2}$$

In the following we often regard $H^0(C, \mathcal{O}_{\mathcal{C}}(* \sum_{j=1}^N Q_j))$ as a subspace of $\bigoplus_{j=1}^N K((\xi_j))$.

Let us fix a non-negative integer ℓ. For each $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$, a left $\widehat{\mathfrak{g}}_N(K)$-module $\mathcal{H}_{\vec{\lambda}}(K)$ and a right $\widehat{\mathfrak{g}}_N(K)$-module $\mathcal{H}^\dagger_{\vec{\lambda}}(K)$ are defined as

$$\mathcal{H}_{\vec{\lambda}}(K) = \mathcal{H}_{\lambda_1}(K) \otimes_K \cdots \otimes_K \mathcal{H}_{\lambda_N}(K)$$
$$\mathcal{H}^\dagger_{\vec{\lambda}}(K) = \mathcal{H}^\dagger_{\lambda_1}(K) \widehat{\otimes}_K \cdots \widehat{\otimes}_K \mathcal{H}^\dagger_{\lambda_N}(K),$$

where $\vec{a} = (X_1 \otimes f_1, \ldots, X_N \otimes f_N)$ acts as

$$\vec{a}|\varphi_1 \otimes \ldots \varphi_N\rangle = \sum_{j=1}^{N} |\varphi_1 \otimes \ldots \otimes (X_j \otimes f_j)\varphi_j \otimes \ldots \otimes \varphi_N\rangle$$

$$\langle\phi_1 \otimes \ldots \phi_N|\vec{a} = \sum_{j=1}^{N} \langle\phi_1 \otimes \ldots \otimes \phi_j(X_j \otimes f_j) \otimes \ldots \otimes \phi_N|$$

Now we are ready to define the space of vacua attached to $\mathfrak{X}_K$.

Definition 2.1. Assume that each irreducible component C_i of the curve C contains at least one Q_j. Put

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}_K) = \mathcal{H}_{\vec{\lambda}}(K)/\widehat{\mathfrak{g}}(\mathfrak{X}_K)\mathcal{H}_{\vec{\lambda}}(K).$$

The K-vector space $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}_K)$ is called the *space of covacua* attached to $\mathfrak{X}_K$. The *space of vacua* or *conformal blocks* attached to $\mathfrak{X}_K$ is defined as

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_K) = \mathrm{Hom}_K(\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}), K).$$

For any field extension K'/K put

$$\mathfrak{X}_{K'} = \mathfrak{X}_K \otimes K'.$$

Then, it is easy to show that

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}_{K'}) = \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}_K) \otimes_K K', \quad \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_{K'}) = \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_K) \otimes_K K'.$$

In particular, we have

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}_{\mathbf{C}}) = \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}_K) \otimes_K \mathbf{C}$$
$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_{\mathbf{C}}) = \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_K) \otimes_K \mathbf{C}.$$

Thus by Theorem 6.2.1 and Corollary 6.2.3 of [TUY] we have the following Theorem.

Theorem 2.2. *$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_K)$ is a finite dimensional K-vector space. Its dimension depends on the genus g of the curve C, the number N of the points Q_j, $\vec{\lambda}$ and the level ℓ.*

For $\mathfrak{X}_K = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$ let P be a non-singular point of the curve C defined over K', a field extension of K and η a formal neighbourhood of C at P defined over K'. Put

$$\widetilde{\mathfrak{X}}_{K'} = (C; Q_1, \ldots, Q_N, P; \eta_1, \ldots, \eta_N, \eta).$$

In the following we fix a highest weight vector $|0\rangle$ of the integrable left $\widehat{\mathfrak{g}}(\mathbf{Q})$-module $\mathcal{H}_0(\mathbf{Q})$. Since there is a canonical inclusion

$$\begin{array}{ccc} \mathcal{H}_{\vec{\lambda}}(\mathbf{Q}) & \longrightarrow & \mathcal{H}_{\vec{\lambda}}(\mathbf{Q}) \otimes \mathcal{H}_0(\mathbf{Q}) \\ |v\rangle & \mapsto & |v\rangle \otimes |0\rangle \end{array}$$

we have a canonical surjection

$$\hat{\imath}^* \ : \ \mathcal{H}^\dagger_{\vec{\lambda}}(\mathbf{Q}) \widehat{\otimes} \mathcal{H}^\dagger_0(\mathbf{Q}) \longrightarrow \mathcal{H}^\dagger_{\vec{\lambda}}(\mathbf{Q}) \ .$$

Theorem 2.3. *The canonical surjection $\hat{\imath}^*$ induces a canonical isomorphism*

$$\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}_{K'}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}_K) \otimes K'.$$

Proof. By Proposition 2.2.3 of [TUY] we have a canonical isomorphism

$$\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}_{K'}) \otimes \mathbf{C} \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}_K) \otimes'_K \mathbf{C}.$$

(Unfortunately the proof of [TUK] contains a stupid mistake, but it is easy to fix it. (See for example [U2].)) Since we have

$$\hat{\imath}^*(\mathcal{V}^\dagger_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}_{K'})) \subset \mathcal{H}^\dagger_{\vec{\lambda}}(\mathfrak{X}_K)(K') \cap \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}_K) \otimes \mathbf{C} = \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}_K) \otimes K',$$

we have the desired result.

Let us assume that for an N-pointed stable curves with formal neighbourhoods $\mathfrak{X}_K = (C; Q_1, Q_2, \ldots, Q_N; \eta_1, \eta_2, \ldots, \eta_N)$ defined over K, the curve C has a double point P defined over a field K', an extension of the field K. Let $\nu : \widetilde{C} \to C$ be a morphism obtained by the normalization of the double point P. Note that $\nu : \widetilde{C} \to C$ is defined over the field K' and the inverse image $\nu^{-1}(P)$ consists of two smooth K'-rational points P', P'' of the curve $\widetilde{C}$. Choose formal neighbourhoods η', η'' of the points P' and P'', respectively and put

$$\widetilde{\mathfrak{X}}_{K'} = (\widetilde{C}; Q_1, \ldots, Q_N, P', P''; \eta_1, \ldots, \eta_N, \eta', \eta'').$$

Then, we have the following factorization theorem.

Theorem 2.4. *Under the above notation, there exists a canonical isomorphism*

$$\bigoplus_{\mu \in P_\ell} \mathcal{V}^\dagger_{\vec{\lambda},\mu,\mu^\dagger}(\widetilde{\mathfrak{X}}_{K'}) \simeq \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}_K) \otimes_K K'. \tag{2.2}$$

When $K = K' = \mathbf{C}$, the theorem is nothing but Proposition 2.2.6 in [TUY]. By a similar argument as in the proof of Theorem 2.3 , we can easily prove Theorem 2.4.

Next let us consider a family of N-pointed stable curves with formal neighbourhoods defined over the field K.

Definition 2.5. Data $(\pi : Y \to B; s_1, s_2, \ldots, s_N; \eta_1, \eta_2, \ldots, \eta_N)$ are called a family of N-pointed stable curves of genus g with formal neighbourhoods defined over K, if the following conditions are satisfied.

(1) Y and B are K-schemes of countable type, $\pi : Y \to B$ is a proper flat morphism and $s_1, s_2, \ldots, s_N$ are sections of π defined over the field K.

(2) For each point $b \in B$ the data $(Y_b := \pi^{-1}(b); s_1(b), s_2(b), \ldots, s_N(b))$ is an N-pointed $K(b)$-stable curve of genus g where $K(b)$ is the field $\mathcal{O}_{\mathcal{B},b}/\mathfrak{m}_b$.

(3) η_j is an $\mathcal{O}_B$-algebra isomorphism

$$\eta_j : \widehat{\mathcal{O}}_{Y/s_j} = \varprojlim_n \mathcal{O}_Y/I_j^n \simeq \mathcal{O}_B[[\xi]],$$

where I_j is the defining ideal of $s_j(B)$ in Y.

Similarly we define a family of N-pointed curves with n-th formal neighbourhoods defined over K. Let $\mathfrak{F}_K = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_n; \eta_1, \ldots, \eta_N)$ be a family of N-pointed stable curves of genus g with formal coordinates defined over the field K. The sheaf $\widehat{\mathfrak{g}}_N(\mathcal{B})$ of affine Lie algebra over $\mathcal{B}$ is a sheaf of $\mathcal{O}_{\mathcal{B}}$-module

$$\widehat{\mathfrak{g}}_N(\mathcal{B}) = \mathfrak{g} \otimes_{\mathbf{C}} (\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j))) \oplus \mathcal{O}_{\mathcal{B}} \cdot c$$

with the following commutation relation.

$$\begin{aligned}[(X_1 \otimes f_1, \ldots, X_N \otimes f_N),&(Y_1 \otimes g_1, \ldots, Y_N \otimes g_N)] \\ &= ([X_1, Y_1] \otimes (f_1 g_1), \ldots, [X_N, Y_N] \otimes (f_N g_N)) \\ &\quad \oplus c \cdot \sum_{j=1}^{N} (X_j, Y_j) \operatorname*{Res}_{\xi_j = 0}((g_j df_j)\end{aligned}$$

$$c \in \text{Center}$$

where

$$X_j,\, Y_j \in \mathfrak{g}(K), \quad f_j,\, g_j \in \mathcal{O}_{\mathcal{B}}((\xi_j))\,.$$

Put

$$\widehat{\mathfrak{g}}(\mathfrak{F}_K) = \mathfrak{g}(K) \otimes_K \pi_*(\mathcal{O}_{\mathcal{C}}(*S))$$

where we define

$$S = \sum_{j=1}^{N} s_j(\mathcal{B})$$

$$\pi_*(\mathcal{O}_{\mathcal{C}}(*S)) = \varinjlim_k \pi_*(\mathcal{O}_{\mathcal{C}}(kS))\,.$$

There is a sheaf version of homomorphism (2.2), by using the formal neighbourhoods η_j:

$$\tilde{t} : \pi_*(\mathcal{O}_{\mathcal{B}}(*S)) \to \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j))$$

and we may regard $\widehat{\mathfrak{g}}(\mathfrak{F}_K)$ as a Lie subalgebra of $\widehat{\mathfrak{g}}_N(\mathcal{B})$.

Fix a non-negative integer ℓ. For any $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$, define

$$\mathcal{H}_{\vec{\lambda}}(\mathcal{B}) = \mathcal{O}_{\mathcal{B}} \otimes_K \mathcal{H}_{\vec{\lambda}}(K)$$
$$\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}) = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{H}_{\vec{\lambda}}(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) = \mathcal{O}_{\mathcal{B}} \otimes_K \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}).$$

The pairing (1.1) induces an $\mathcal{O}_{\mathcal{B}}$-bilinear pairing

$$\langle \quad | \quad \rangle : \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}) \times \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) \to \mathcal{O}_{\mathcal{B}} \tag{2.3}$$

which is complete with respect to the filtration defined in §1. The sheaf of affine Lie algebra $\widehat{\mathfrak{g}}_N(\mathcal{B})$ acts on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ by

$$((X_1 \otimes \sum_{n\in\mathbf{Z}} a_n^{(1)} {\xi_1}^n), \dots, (X_N \otimes \sum_{n\in\mathbf{Z}} a_n^{(N)} {\xi_N}^n))(F \otimes |\Psi\rangle)$$
$$= \sum_{j=1}^{N} \sum_{n\in\mathbf{Z}} (a_n F) \otimes \rho_j(X_j(n))|\Psi\rangle,$$

where $\rho_j(X_j(n))$ means that $X_j(n)$ acts on j-th component of $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$.

The action of $\widehat{\mathfrak{g}}_N(\mathcal{B})$ on $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ is the dual action of $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$, that is,

$$\langle \Psi a | \Phi \rangle = \langle \Psi | a\Phi \rangle \quad \text{for any} \quad a \in \widehat{\mathfrak{g}}_N(\mathcal{B}).$$

Definition 2.6. Put

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_K) = \mathcal{H}_{\vec{\lambda}}(\mathcal{B}) / \widehat{\mathfrak{g}}(\mathfrak{F}_K)\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$$
$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_K) = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{B}}}(\mathcal{V}_{\vec{\lambda}}(\mathcal{B}), \mathcal{O}_{\mathcal{B}}).$$

These are sheaves of $\mathcal{O}_{\mathcal{B}}$-modules on $\mathcal{B}$. The sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_K)$ is called the *sheaf of vacua* or *conformal blocks* attached to the family $\mathfrak{F}_K$ and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_K)$ is called a *sheaf of covacua*. Note that we have

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_K) = \{\, \langle \Psi | \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B}) \,|\, \langle \Psi | a = 0 \quad \text{for any} \quad a \in \widehat{\mathfrak{g}}(\mathfrak{F}_K) \,\}.$$

The pairing (2.3) induces a non-degenerate $\mathcal{O}_{\mathcal{B}}$-bilinear pairing

$$\langle \quad | \quad \rangle : \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_K) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_K) \to \mathcal{O}_{\mathcal{B}}.$$

For each field extension K'/K put

$$\mathfrak{F}_{K'} = \mathfrak{F} \otimes_K K'.$$

Then, we have

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_{K'}) = \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}) \otimes_K K', \quad \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_{K'}) = \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_K) \otimes_K K'.$$

Let Y be a K-scheme of finite type and $\mu : Y \to \mathcal{B}$ be a K-morphism. We let $\mathfrak{F}_Y$ be the pull-back of the family $\mathfrak{F}_K$ by the morphism μ. The results in [TUY] imply the following theorem.

Theorem 2.7. *We have*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_Y) = \mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_K) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_Y, \quad \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_Y) = \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_K) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_Y$$

and $\mathcal{V}_{\vec{\lambda}}(\mathfrak{F}_Y)$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_Y)$ are locally free $\mathcal{O}_Y$-modules of finite rank.

§3 Projectively flat connection

Let $\mathfrak{F}_K = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_n; \eta_1, \ldots, \eta_N)$ be a universal family of N-pointed stable curves of genus g with formal coordinates defined over a field K of characteristic zero. Hence, the base scheme $\mathcal{B}$ is an infinite dimensional K-scheme. There exists an exact sequence of $\mathcal{O}_{\mathcal{B}}$-modules

$$0 \to \pi_* \Theta_{\mathcal{C}/\mathcal{B}}(*S) \to \bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j)) \frac{d}{d\xi_j} \xrightarrow{\theta} \Theta_{\mathcal{B}}(-\log D) \to 0$$

where D is the locus in $\mathcal{B}$ of singular curves and θ is given by the Kodaira-Spencer mapping. We can define the action of $\displaystyle\bigoplus_{j=1}^{N} \mathcal{O}_{\mathcal{B}}((\xi_j)) \frac{d}{d\xi_j}$ on $\mathcal{H}^{\dagger}_{\vec{\lambda}}(\mathcal{B})$ by

$$D(\vec{\ell})(F \otimes \langle \Psi |) = \theta(\vec{\ell})(F) \otimes \langle \Psi | + \sum_{j=1}^{N} F(\langle \Psi | \rho_j(T[\underline{\ell}_j])$$

where

$$F \in \mathcal{O}_{\mathcal{B}} \quad \langle \Psi | \in \mathcal{H}^{\dagger}_{\vec{\lambda}}(K), \quad \vec{\ell} = (\underline{\ell}_j) \in \oplus \mathcal{O}_{\mathcal{B}}((\xi_j)) \frac{d}{d\xi_j},$$

for $\underline{\ell}_j = \ell_j \dfrac{d}{d\xi_j} \in \mathcal{O}_{\mathcal{B}}((\xi_j))\dfrac{d}{d\xi_j}$, $T[\underline{\ell}_j]$ is defined by

$$\operatorname*{Res}_{\xi_j=0}(T(\xi_j)\ell_j d\xi_j)$$

and ρ_j means that $T[\underline{\ell}_j]$ acts on the j-th factor of $\mathcal{H}^{\dagger}_{\vec{\lambda}}(K)$. Then, this action induces the action on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{F}_K)$ which defines the projectively flat connection with regular singularities along D. Thus the projective flat connection is defined over the filed K. Note that the moduli space $\overline{\mathfrak{M}}_{g,N}$ of N-pointed stable curves of genus g with formal neighbourhoods exists as an infinite-dimensional scheme defined over the rational numbers $\mathbf{Q}$. Hence, the projective connection of the sheaf of vacua is defined over $\mathbf{Q}$. To descend the theory to finite-dimensional situation we cannot apply the same procedure as in [TUK, §5], since the structure of $Lie(\mathrm{Aut}(\mathbf{Q}[[\xi]])$ is complicated as a Lie algebra over $\mathbf{Q}$. Instead we first consider the moduli stack of N-pointed stable curves of genus g with the first order neighbourhoods. Locally we construct a family $\mathfrak{F}_{\mathbf{Q}} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_N; \eta_1, \ldots, \eta_N)$ of N-pointed stable curves of genus g with the formal neighbourhoods over $\mathbf{Q}$ which induces a versal family of N-pointed stable curves of genus g with the first order neighbourhoods. Then, on $\mathfrak{F}_{\mathbf{Q}}$ we can introduce a projectively flat connection similar to above. (For details, see, for example, [U2].) Note that the connection thus constructed is compatible with respect to the restriction of the family of N-pointed stable curves with formal neighbourhoods. Also for N-pointed stable curves with formal neighbourhoods defined over K $\mathfrak{X}_K$ and $\mathfrak{X}'_K$ which induce isomorphic N-pointed stable curves with the first order neighbourhoods the spaces of vacua $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_K)$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}'_K)$ are canonically isomorphic as K-vector spaces, since $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_K) \otimes \mathbf{C}$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}'_K) \otimes \mathbf{C}$ are canonically isomorphic. In this way we can define the sheaf of vacua on the moduli stack of N-pointed stable curves with the first order neighbourhoods and the projectively flat connection on it.

Remark. The monodromy representation of the connection is interesting. For example, for $g = 0$ it is deeply related to representations of braid groups. Moreover, the result of [TK] suggests that the monodromy representations are defined over a cyclotomic number field.

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